A Numerical Analytical Model for Stochastic Dynamic User-Optimal Route Choice

M. Jia

Center for Traffic and Transport, Technical University of Denmark
DK-2800 Kgs. Lyngby, Denmark

Abstract: Dynamic traffic assignment (DTA) models can be realized with vehicle-based simulation approach or flow-based analytical approach. Depending on methodology, the flow-based analytical models can be further classified into mathematical programming, optimal control and variational inequality models. A brief review of the development of analytical models is given in this paper. A stochastic dynamic user optimal (SDUO) route choice model is formulated with variational inequality (VI) approach. A nested diagonalized algorithm is adopted to solve the SDUO route choice model. After relaxation method, the diagonalized subproblem, which is a nonlinear programming problem (NLP), can be solved with Method of Successive (MSA). The MSA method is consists of a LP subproblem. In order to avoid route enumeration, a dynamic variation of STOCH method is adopted to solve this multinomial logit model. A numerical example is given to verify the solution of the proposed algorithm.

Key Words: Dynamic assignment; Dynamic user optimal; Variational Inequality; Stochastic

\textsuperscript{1}E-mail: mj@ctt.dtu.dk
1 Introduction

Dynamic traffic assignment (DTA) models have been studied intensively since Merchant and Nemhauser [14,15] formulated a dynamic system-optimal (DSO) route choice model in 1978. DTA can be realized with vehicle-based simulation approach or flow-based analytical approach. Over the past twenty years, the analytical models for dynamic traffic assignment received a lot of attention from researchers and practitioners. Depending on methodology employed, the analytical DTA models can be further classified into mathematical programming, optimal control and variational inequality models.

With the advantages of performing qualitative analysis and handling more realistic traffic scenarios, Variational inequality approach is applied in many traffic assignment models especially with asymmetric link interactions. It is more general than the other analytical models and as a result, recent analytical DTA models have migrated towards the variational inequality approach. The variational inequality model introduced in this paper is a stochastic dynamic user optimal route choice model with discrete time, predictive traffic information, which is based on the research work of Ran, Boyce, (1996) [18] and Chen, H.K., (1998) [5,6].

The remainder of the paper is organized as follows. In the next section, a literature review of the analytical DTA models is presented. In section 3, the important definition and theorems related to variational inequality problem formulation is reviewed. Then, the stochastic dynamic user-optimal route choice model are introduced including model formulation and algorithm. A numerical example is given to verify the solution of the proposed algorithm. Finally, some conclusions are presented in the last section.

2 Literature Review of Analytical DTA Models

2.1 Mathematical Programming Models

Merchant and Nemhauser (1978a,b) [14,15] are the first to formulate the DTA problem as a mathematical program. They presented a deterministic, fixed-demand, discontinuous, nonlinear, and non-convex dynamic system-optimal (DSO) traffic assignment model (M-N model) for a many-to-one network. The global solution is obtained by solving a piecewise linear version of the model.

Ho (1980) [9] solved the global optimum of M-N model by successively optimising a sequence of at most N+1 linear programs, where N is the number of periods and derived sufficient condition for the optimal solution.

Janson (1991a,b), Janson and Robles (1995) [10–12] presented a set of dynamic network models in terms of experienced path travel times instead of instantaneous travel times and proposed a heuristic solution algorithm. It is one of the earliest to formulate UE DTA problem as a mathematical program.


### 2.2 Optimal Control Models

Luque and Friesz (1980) [3] formulated the first DSO problem using optimal control theory. Later, Friesz et al. (1989) [7] discussed link-based optimal control formulation for both the DSO and DUO problems. They treat the link inflow as a control variable through the exit flow is treated as a nonlinear function. The problems include the lack of meaningful link performance functions and exit functions, and it also has computational difficulties.

Ran and Shimazaki (1989a) [19] use the optimal control approach to develop a link-based SO model with multiple origins and destinations. Ran and Shimazaki (1989b) [20] presented an optimal control theory based instantaneous UE DTA model. They treated the exit flow as a variable rather than function and formulate a many-to-may dynamic network models. Later, Ran and Boyce (1994) [1] developed the solution algorithm for such an instantaneous UE DTO model.
2.3 Variational Inequality (VI) Models

Many complex transportation network problems have been formulated with variational inequality. Some dynamic route choice models realized with VI approach are briefly summarized in the following Table 1.

**Table 1: Dynamic Route Choice Models with Variational Inequality Approach**

<table>
<thead>
<tr>
<th>Route Choice Models</th>
<th>Characteristics</th>
<th>Some Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Friesz et al., 1993 [8]</td>
<td>Continuous time, infinite-dimensional VI model</td>
<td>Departure time/route choice</td>
</tr>
<tr>
<td>Wie et al., 1994 [21]</td>
<td>Discretized time VI model</td>
<td>Simultaneous route – departure time equilibrium</td>
</tr>
<tr>
<td>Ran, Boyce, 1996 [18]</td>
<td>A set of both link-based and route-based discretized VI models with fixed departure times or simultaneous departure time etc.</td>
<td>Including queuing delaying</td>
</tr>
</tbody>
</table>
2.4 Summary

A comparison of the above three approaches is listed in Table 2.

<table>
<thead>
<tr>
<th>Approaches</th>
<th>Advantages</th>
<th>Limitations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mathematical programming</td>
<td>Simple in computation</td>
<td>With the difficulties to represent various realistic situations</td>
</tr>
<tr>
<td>Optimal Control</td>
<td>Better for describing dynamic systems having variables associated with continues time</td>
<td>Suffer some difficulties for representing various realistic situations as mathematical programming approach</td>
</tr>
<tr>
<td>Variational Inequality</td>
<td>Better analytical qualitative analysis and easy to formulate various DTA problems; Better represent link interactions, especially with asymmetric Jacobian matrices for the travel cost functions.</td>
<td>Require intensive computations;</td>
</tr>
</tbody>
</table>

3 The Stochastic Dynamic User-Optimal Route Choice Model

3.1 Variational Inequality Problem

Variational inequality provides a general formulation platform including a set of mathematical problems, such as optimisation problems, fixed-point problem, system of equations and complementarity. Some of the most important definitions and theorems are listed here as a quick reference. The comprehensive summary of VI can be found at Nagurney (1998) [16].

**Definition (Variational Inequality Problem):** The finite-dimensional variational inequality problem $VI(,G)$, is to determine a vector $^* \in G \subset R^n$, such that

$$
\left[ f^* \right] \left[ -^* \right] \geq 0, \ \forall \ \in G, \text{ where } f \text{ is a given continuous function from } G \text{ to } R^n \text{ and } G \text{ is a given closed convex set.}
$$
Both unconstrained and constrained optimisation problems can be formulated as variational inequality problems. The following two theorems set the relationship between an optimisation problem and a variational inequality problem.

**Theorem 1:** Let $x^*$ be a solution to the optimisation problem:
$$\min_{x \in G} Z(x),$$
where $Z$ continuously differentiable and $G$ is closed and convex. Then, $x^*$ is a solution of the variational inequality problem $VI(\nabla Z, G)$:
$$\nabla Z(x^*) \cdot (u - x) \geq 0, \forall u \in G.$$

**Theorem 2:** If $Z(x)$ is a convex function and $x^*$ is a solution to variational inequality $VI(\nabla Z, G)$, then $x^*$ is a solution of the optimization problem:
$$\min_{x \in G} Z(x).$$

On the other hand, the variational inequality problem can be formulated as an optimisation problem only when a certain symmetry condition hold true for the given function. It is depicted in the following theorem.

**Theorem 3:** If $F(x)$ is continuously differentiable on $G$ and the Jacobian matrix
$$\nabla F(x) = \begin{bmatrix}
\frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_n} \\
\frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \cdots & \frac{\partial F_n}{\partial x_n}
\end{bmatrix}$$

is symmetric and positive semi-definite. Then there is a real-valued convex function $Z(x)$ satisfying
$$\nabla Z(x) = (x - x^*),$$
with $x^*$ the solution of the variational inequality $VI(\nabla Z, G)$, where $G$ is closed and convex, is also the solution to the optimization problem:
$$\min_{x \in G} Z(x).$$
3.2 Stochastic Dynamic User-Optimal Route (SDUO) Choice Model Formulation

The variational inequality approach has been proved very useful for its relation to an equivalent optimisation formulation. It is a more general approach since it can accommodate a function with an asymmetric Jacobian matrix. It is practically important for dynamic travel choice model, because the Jacobian matrix of the travel time function is usually asymmetric, vehicles can be affected by the previously entered flow while the converse is not true. Thus the equivalent optimisation problem cannot be obtained. Only by temporarily fix the actual link travel time and flow interaction in different time unit at current level, the dynamic travel choice models can be reduced to an optimisation subproblem. Both Ran, Boyce, (1996) [18] and Chen, H.K., (1998) [5,6] have formulated a set of stochastic dynamic user-optimal route choice models with variational inequality approach, the following model and algorithms are mainly based on their research.

3.2.1 Notations

Throughout this formulation, the superscript $rs$ denotes the origin-destination pair from $r$ to $s$, the subscript $a$ denotes link $a$ and the subscript $p$ denotes path $p$. The variables used in the formulation include the following:

- $\hat{c}_{rs}^p$, the perceived travel time for route $p$ between O-D pair $rs$;
- $\tau_a(t)$, the estimated actual link travel times;
- $h_p^r(k)$, the flow over route $p$ departing during interval $k$;
- $\hat{r}_{rs}^r(k)$, the minimal perceived route travel time between O-D pair $rs$ during time interval $k$;
- $\hat{c}_{rs}^p(k)$, the perceived travel time for route $p$ between O-D pair $rs$ during time interval $k$;
- $\delta_{apk}(t) = 1$, if inflow rate on link $a$ during time interval $t$ departs from origin $r$ over route $p$ towards destination $s$ during time interval $k$, otherwise 0;
- $\bar{q}_{rs}(k)$, the fixed departure flow between O-D pair $rs$ during time interval $k$;
- $u_a(t)$, inflow into link $a$ during time interval $t$;
- $t$, entering interval for a link.
- $k$, departure interval for a route.
- $*$ = at equilibrium conditions.

3.2.2 SDUO Conditions

The stochastic dynamic user-optimal conditions are defined as:

For each O-D pair that the perceived route travel times for travellers departing during the same interval are equal to the minimal perceived route travel time; no
traveller would be better off with respect to the perceived route travel times by unilaterally changing his/her route.

The assumptions of this model are the available traffic information is perfect to all travellers and the route choice for travellers is pre-trip minimal actual route travel time. No route switching is permitted en route. So, it is a predictive stochastic dynamic user optimal model.

These equilibrium conditions can be mathematically expressed as follows:

\[
\hat{c}^r_p(k) = \hat{\pi}^r(k) \text{ if } h^r_p(k) > 0 \\
\hat{c}^r_p(k) \geq \hat{\pi}^r(k) \text{ if } h^r_p(k) = 0 \quad \forall r, s, p, k
\]  

(3-1)

where \( \hat{c}^r_p(k) \) is the perceived travel time for route \( p \) between O-D pair \( rs \) during time interval \( k \) with two components: a systematic term and error term.

### 3.2.3 Network Constraints

The constraints set for the stochastic dynamic route choice problem is as follows:

**Flow conservation constraint:**

\[
\sum_p h^r_p(k) = \bar{q}^r(k) \quad \forall r, s, k
\]  

(3-2)

**Flow propagation constraints:**

\[
u^r_{apk}(t) = h^r_p(k) \delta^r_{apk}(t) \quad \forall r, s, a, p, k, t
\]

\[
\sum_t \delta^r_{apk}(t) = 1 \quad \forall r, s, a, p, k
\]

\[
\delta^r_{apk}(t) = [0,1] \quad \forall r, s, a, p, k, t
\]

(3-3)

**Nonnegativity constraints:**

\[
h^r_p(k) \geq 0 \quad \forall r, s, p, k
\]  

(3-4)

**Definitional constraints:**

\[
u_a(t) = \sum_p \sum_k h^r_p(k) \delta^r_{apk} \quad \forall a, t
\]

\[
c^r_p(k) = \sum_a \sum_t c_a(t) \delta^r_{apk} \quad \forall r, s, p, k
\]

\[
\hat{c}^r_p(k) = c^r_p(k) + \frac{1}{\theta} \ln h^r_p(k) \quad \forall r, s, p, k
\]  

(3-5)
3.2.4 Variational Inequality Problem

The SDUO route choice problem is equivalent to finding a solution \( u^* \in \Omega \) such that the following VIP holds:

\[
\hat{u}^* \geq \sum \left[ - h_p^r(k) - h_p^s(k) \right] \quad \forall \in \Omega^*
\]

Or in the expanded form:

\[
\sum_{r} \sum_{p} \sum_{k} \hat{c}_{pk}^r(k) \left( h_p^r(k) - h_p^s(k) \right) \geq 0 \quad \forall h \in \Omega^*
\]  (3-6)

where \( \Omega^* \) is a subset of \( \Omega \) with \( \delta_{apk}^{rs}(t) \) being realized at equilibrium, i.e. \( \delta_{apk}^{rs}(t) = \delta_{apk}^{rs}(t) \), \( \forall r,s,a,p,k,t \), \( \Omega \) denotes the feasible region that subject to the network constraints shown as (3-2) - (3-5).

With a certain flow propagation relationship, \( \delta_{apk}^{rs}(t) = \delta_{apk}^{rs}(t) \), the SDUO route choice conditions (3-1) will infer VIP (3-6) and vice versa. The equivalence analysis between equilibrium condition (3-1) and the proposed VIP (3-6) can be found in [6].

3.3 Solution Algorithm

The general iterative scheme for the solution of the variational inequality problem is introduced in [16], which contains, as special cases, the projection, linearization, and relaxation methods. The relaxation method (also called as diagonalization method) solves variational inequality into a sequence of subproblems, which are in general, nonlinear programming problems (NLP).

For the SDUO route choice model, two kinds interactions have been considered. One is the actual link travel time, which is unknown in advance. Based on a given feasible flow pattern, the actual link travel time can be estimated and the corresponding new subproblem’s feasible region can be determined. Another one is the well known temporal interactions between flows enter the same physical link. The inflows can be affected by those previously entered flow. The Jacobian matrix of dynamic travel time function is therefore asymmetric. By temporarily fixing the previously entered flows in the same physical link, this link interaction can be relaxed.

A nested diagonalized algorithm is adopted to solve the SDUO route choice. After relaxation method, the diagonalized subproblem is generated. It is a nonlinear programming problem (NLP) which can be solved with Method of Successive (MSA). The nested diagonalization algorithm for SDUO route choice model is as follows:
Nested Diagonalization Algorithm for SDUO Route Choice Model

Step 0: Initialisation
Step 0.1: let \( m=0 \), set \( \tau_a^0(t) = NINT\left[C_{a,t}(t)\right] \forall a,t \)
Step 0.2: let \( n=1 \), find an initial feasible solution \( \{\mu_a^1(t)\} \). Compute the associated link travel times \( \{c_a^1(t)\} \).

Step 1: First loop (Relaxation, Outer iteration)
Let \( m=m+1 \), update the estimated actual link travel times by
\[
\tau_a^m(t) = NINT\left[(1-\gamma)\tau_a^{m-1}(t) + \gamma c_a^m(t)\right] \forall a,t
\]
construct the feasible time-space network based on the estimated actual link travel times accordingly.

Step 2: Second loop (Relaxation, Second iteration)
Step 2.1: let \( n=1 \). Compute and reset the initial feasible solution \( \{\mu_a^n(t)\} \), based on the time-space network constructed by the estimated actual link travel times \( \{\tau_a^m(t)\} \).
Step 2.2: Fix the inflows for all time-space links other than on the subject time-space link at the current level, yielding the NLP problem:
\[
\min z(u,\theta) = \sum_a \sum_t \int_0^{u_a^m(t)} c_a(0,\mu_a^n(t),u_a^n(t),\theta)dt + \sum_a \sum_p \sum_k \left[ h_p^m(k)\ln(h_p^n(k)) - h_p^n(k) \right]
\]
Subject to:
network constraints (3-2)\textendash(3-5).

Step 3: Third loop (Inner iteration)
Solve for the solution, \( \{\mu_a^{n+1}(t)\} \), in the NLP problem by the Method of Successive Averages (MSA), compute the resulting link travel times \( \{c_a^{n+1}(t)\} \).

Step 4: Convergence check for the second loop operation
If \( u_a^{n+1}(t) \approx u_a^n(t) \forall a,t \), go to Step 5; otherwise, set \( n=n+1 \), go to Step 2.2.

Step 5: Convergence check for the first loop operation
If \( \tau_a^m(t) \approx c_a^{n+1}(t) \forall a,t \), stop; the current solution is optimal. Otherwise, set \( n=n+1 \), and go to step 1.

At step 2, the NLP problem can be solved by using Method of Successive Average (MSA). In the dynamic case, the MSA algorithm is as follows:
Step 1:
update link travel time based on the previous feasible solution \( \{u_a^n(t)\} \):
\[
c_a(t) = c_a(u_a^n(1), u_a^n(2), ..., u_a^n(t)) \quad \forall a, t
\]

Step 2:
Solve for the following LP problem, yielding the inflows \( \{p_a^n(t)\} \).

\[
\min z(w) = \sum_{rs} \sum_{p} \sum_{k} \left[ c_a^n(k)^n + \frac{1}{\theta} \ln(h_{p}^n(k)^n) \right] w_{p}^n(k)
\]

Subject to:
Flow conservation constraint:
\[
\sum_{p} w_{p}^n(k) = \bar{q}^n(k) \quad \forall r, s, k
\]
Nonnegativity constraints:
\[
w_{p}^n(k) \geq 0 \quad \forall r, s, p, k
\]
Definitional constraints:
\[
p_{ap}^n(t) = w_{p}^n(k)\delta_{ap}^n(t) \quad \forall r, s, a, p, k, t
\]
\[
\delta_{ap}^n(t) = \{0, 1\} \quad \forall r, s, a, p, k, t
\]
\[
p_a(t) = \sum_{rs} \sum_{p} \sum_{k} w_{p}^n(k)\delta_{ap}^n(t) \quad \forall a, t
\]
\[
c_p^n(t) = \sum_{a} c_a(t)\delta_{ap}^n(t) \quad \forall r, s, p, k
\]
\[
\check{c}_p^n(k) = c_p^n(k) + \frac{1}{\theta} \ln w_{p}^n(k) \quad \forall r, s, p, k
\]

Step 3:
Update the flow pattern:
\[
u_a^{n+1} = u_a^n(t) + \lambda^n(p_a^n(t) - u_a^n(t)) \quad \forall a, t
\]
\[
\lambda^n = \frac{l_1}{l_2 + n}
\]
where \( l_1 \) is a positive constant and \( l_2 \) is a nonnegative number.

Step 4: if convergence criterion is met, stop. Otherwise, set \( n = n + 1 \), and go to step 1.

Theoretically, the LP problem in Step 2 of MSA can be solved by route enumeration. However, such method is prohibited for a large networks. Ran, Boyce, (1996) [18] and Chen, H.K (1998) [6] have suggested a link-based heuristic procedure to solve this problem. It a dynamic variation of STOCH method namely DYNASTOCH1 and SADA with the main proceeding difference of backward pass vs. forward pass. Each method starts from identifying
the set of efficient routes connecting each O-D pair, then assigns probabilities and flows to
efficient route connecting each O-D pair for each time interval.

In Step 2 of MSA method, the following SADA algorithm is performed repeatedly for each
O-D pair \( rs \) in the network. Given the updated link travel time \( c_a(t) = c_a(u_a^1, u_a^2, \ldots, u_a^n(t)) \)
\( \forall a, t \), and suppose link \( a \) is alternatively represented by a tail node \( i(t) \) and a head node \( j \),
i.e. \( a = (i(t), j) \).

Step 0: Initialization

Compute the minimum perceived travel time \( \pi_{r(t)}^r(k) \) from node \( r \) to all other nodes \( i \).
Define \( m \in \beta_i(t) \) as the set of upstream nodes of all links arriving at node \( i \) during
interval \( t \).
Define \( n \in \alpha_i(t) \) as the set of downstream nodes of all links leaving at node \( i \) during
interval \( t \).

Calculate the likelihood, \( L_{rjs}^n(k) \) for each link \( ij \) during each interval \( t \):

\[
L_{rjs}^n(t) = e^{\theta_k r(t) - \pi_{r(t)}^r(k)} \text{ if } \pi_{r(t)}^r(k) \leq \pi_{i(t)}^r(k) \text{ and } \pi_{s(t)}^r(t + \tau_{i(t)}^r(k)) > \pi_{r(t)}^r(k + \tau_{r(t)}^r(k))
\]

where \( t = k + \tau_{r(t)}^r(k) \)

Step 1: Forward pass

Starting from the origin \( r \), by examining all node \( i \) in ascending sequence with respect
to \( \pi_{r(t)}^r(k) \), calculate the link weight, \( w_{rjs}^n(t) \), for each \( j \) as follows:

\[
w_{rjs}^n(t) = L_{rjs}^n(t) \text{ if } i = r
\]

\[
w_{rjs}^n(t) = L_{rjs}^n(t) \sum_{m \in \beta_i(t)} w_{mjs}^n(t) \text{ otherwise}
\]

where \( t = t' + \tau_{m(t)}^m(t) \)

Step 2: Backward pass

Consider nodes in descending values of \( \pi_{s(t)}^r(t) \), starting from the destination \( s \), when
each node, \( j \), is considered, compute the link flow \( u_{rjs}^n(t) \) for each \( i \), by following the
formula:
\[ u_{ij,k}^{rs}(t) = \frac{q}{\sum_{\mu \in \mathcal{P}_r} w_{ij,k}^{rs}(t)} \quad \text{for } j = s \]

\[ u_{ij,k}^{rs}(t) = \left[ \sum_{\mu \in \mathcal{P}_r} u_{m,k}^{rs}(\tau^*) \right] \frac{w_{ij,k}^{rs}(t)}{\sum_{\mu \in \mathcal{P}_r} w_{m,k}^{rs}(t)} \quad \text{for all other links} \]

where \( \tau^* = \tau + \tau_{ij}^{rs}(t) \)

Step 3: Compute the inflow rate entering link \( a \) during interval \( t \) by the following formula: \( u_a(t) = \sum_{rs} \sum_k u_{a,k}^{rs}(t) \) the result is renamed as \( p_a^*(t) \) and inserted into Step 2 of the MSA method.

The flow generated by this algorithm is equivalent to a logit-based route assignment between each O-D pair given only reasonable routes are considered.

The overall nested diagonalization algorithm flow chart is shown in Figure 1.
Initialisation:
Calculate Free-flow travel time $C_{a_0}(t)$ and Set $\tau^0_a(t) = NINT\left[ C_{a_0}(t) \right] \forall a,t$
Find an initial flow $\left\{ u^0_a(t) \right\}$ from All-or-Nothing (AON) assignment; Compute the associated link travel times $\left\{ t^0_a(t) \right\}$.

Update the estimated actual link travel times by
$$\tau^*_a(t) = NINT\left[ (1 - \gamma)\tau^{n-1}_a(t) + p^*_a(t) \right] \forall a,t$$
Construct the corresponding feasible time-space network accordingly.

Compute and reset the initial feasible solution $\left\{ u^*_a(t) \right\}$, based on the time-space network constructed by the estimated actual link travel times $\left\{ t^*_a(t) \right\}$.

Fix the inflows for all time-space links other than that on the subject time-space link at the current level.

Calculate $u^{n+1}_a(t)$ by MSA, set the inner iteration counter $n=1$.
1. UPDATE: link travel time $c_a(t) = c_a(u^*_a(t), u^*_a(2), ..., u^*_a(t)) \forall a,t$
2. DIRECTION FINDING: Calculate inflows $\left\{ p^*_a(t) \right\}$ by link-based heuristic procedure
3. MOVE: Update the flow pattern:
   $$u^{n+1}_a(t) = u^*_a(t) + \lambda^*(p^*_a(t) - u^*_a(t)) \forall a,t$$
   where $\lambda_1$ is a positive constant and $\lambda_2$ is a nonnegative number.
4. CONVERGENCE TEST
   - $u^{n+1}_a(t) \approx u^*_a(t) \forall a,t$
     - Yes
     - No, set $n=n+1$.
   - $\tau^{n+1}_a(t) \approx \tau^*_a(t) \forall a,t$
     - Yes
     - No, set $n=n+1$.

Figure 1: Nested Diagonalization algorithm flow chart
### 3.4 Numerical Example for Testing

The test network is shown as in Figure 2, which consists of 9 links and 6 nodes. Node 0 is the origin, and node 5 is the destination and all other nodes are intermediates.

![Test Network Diagram](image)

**Figure 2**: Test Network

The dynamic travel time function is arbitrarily defined as follows:

\[ C_a(t) = b + \alpha(\text{inf low}_a(t))^2 + \beta(\text{onflow}_a(t))^2 \]

Where \( b \)'s value is given as in the figure 1, and \( \alpha \) and \( \beta \) is 0.01 for all links. The assumed origin-destination (OD) departure flow rate is 30, and the departure time interval is \( t = 0 \).

The obtained result is shown in the following Figure 3 and summarized in Table 3.

![Results Diagram](image)

**Figure 3**: Results for Test Network (\( \theta = 1 \))
Table 3: Results for Test Network ($\Theta=1$)

<table>
<thead>
<tr>
<th>Link Time Interval</th>
<th>Entering Time Interval</th>
<th>Inflow</th>
<th>Exit Flow</th>
<th>Number of Vehicles (Onflow)</th>
<th>Link Travel Time</th>
<th>Exiting Time Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 (0—1)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4.21</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>2~3</td>
<td>14.85</td>
<td>0</td>
<td>14.85</td>
<td>4.21</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>14.85</td>
<td>4.21</td>
<td>-</td>
</tr>
<tr>
<td>5 (1—4)</td>
<td>4</td>
<td>5.68</td>
<td>0</td>
<td>5.68</td>
<td>3.32</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>5~6</td>
<td>0</td>
<td>0</td>
<td>5.68</td>
<td>3.32</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>0</td>
<td>0</td>
<td>5.68</td>
<td>3.32</td>
<td>-</td>
</tr>
<tr>
<td>8 (4—5)</td>
<td>7</td>
<td>5.68</td>
<td>0</td>
<td>5.68</td>
<td>4.32</td>
<td>11</td>
</tr>
<tr>
<td></td>
<td>8~10</td>
<td>0</td>
<td>0</td>
<td>5.68</td>
<td>4.32</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>11</td>
<td>0</td>
<td>0</td>
<td>5.68</td>
<td>4.32</td>
<td>-</td>
</tr>
<tr>
<td>4 (0—3)</td>
<td>0</td>
<td>6.99</td>
<td>0</td>
<td>6.99</td>
<td>6.49</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>1~5</td>
<td>0</td>
<td>0</td>
<td>6.99</td>
<td>6.49</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>6.99</td>
<td>6.49</td>
<td>-</td>
</tr>
<tr>
<td>2 (0—2)</td>
<td>0</td>
<td>8.16</td>
<td>0</td>
<td>8.16</td>
<td>3.66</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>1~2</td>
<td>0</td>
<td>0</td>
<td>8.16</td>
<td>3.66</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>8.16</td>
<td>3.66</td>
<td>-</td>
</tr>
<tr>
<td>3 (2—3)</td>
<td>3</td>
<td>8.16</td>
<td>0</td>
<td>8.16</td>
<td>2.67</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>8.16</td>
<td>2.67</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>8.16</td>
<td>2.67</td>
<td>-</td>
</tr>
</tbody>
</table>
By summing up the actual link travel time consisting of each path from node 0 to 5 departing from the interval \( t = 0 \), and the corresponding error term, the resulting perceived route travel times should satisfy the Stochastic Dynamic User Optimal (SDUO) condition.

For instance, considering about the route \( 0 \rightarrow 1 \rightarrow 4 \rightarrow 5 \) departing from during interval 0, then the perceived route travel time can be obtained by summing up the actual link travel time on link \( 0 \rightarrow 1 \) during interval 0, and the actual link travel time on link \( 1 \rightarrow 4 \) during interval 0, and the actual link travel time on link \( 4 \rightarrow 5 \) during interval 0 + \( c_{0 \rightarrow 1}(0) \) and the actual link travel time on link \( 4 \rightarrow 5 \) during interval 0 + \( c_{0 \rightarrow 1}(0) + c_{1 \rightarrow 4}(0 + c_{0 \rightarrow 1}(0)) \), and the error term \( \frac{1}{T} \ln(\inf_{0 \rightarrow 1,5} \theta_{0 \rightarrow 5}) \), thus the perceived link travel time is:

\[
C_{0 \rightarrow 1 \rightarrow 4 \rightarrow 5} = 0 + c_{0 \rightarrow 1}(0) + c_{1 \rightarrow 4}(0 + c_{0 \rightarrow 1}(0)) + c_{4 \rightarrow 5}(0 + c_{0 \rightarrow 1}(0) + c_{1 \rightarrow 4}(0 + c_{0 \rightarrow 1}(0))) + \frac{1}{T} \ln(\inf_{0 \rightarrow 1,5} \theta_{0 \rightarrow 5})
\]

\[
\approx 4.21 + c_{1 \rightarrow 4}(4) + c_{4 \rightarrow 5}(0 + c_{0 \rightarrow 1}(0) + c_{1 \rightarrow 4}(0 + c_{0 \rightarrow 1}(0))) + \frac{1}{T} \ln(\inf_{0 \rightarrow 1,5} \theta_{0 \rightarrow 5})
\]

\[
\approx 4.21 + 3.32 + c_{4 \rightarrow 5}(7) + \frac{1}{T} \ln(\inf_{0 \rightarrow 1,5} \theta_{0 \rightarrow 5})
\]

\[
\approx 4.21 + 3.32 + 4.32 + \frac{1}{T} \ln(5.68)
\]

\[
\approx 13.59
\]

The results are showed in Table 4. The trips departing from the same origin during the same time interval have approximately the same perceived route travel time.
The traffic information will be more certain as the value $\theta$ of gets larger, so when the dispersion parameter $\theta$ is set with a very large value, the Stochastic Dynamic User Optimal (SDUO) route choice model can be reduced to the deterministic route choice model. As a comparison, the deterministic counterpart approximation for the test network is shown in Figure 4 and Table 5.

**Figure 4**: Results for Test Network ($\theta=300$)

**Table 4**: Stochastic Route Travel Times Results for Test Network ($\theta=1$)

<table>
<thead>
<tr>
<th>Route</th>
<th>Flow</th>
<th>Perceived Travel Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-&gt;1-&gt;4-&gt;5</td>
<td>5.68</td>
<td>13.59</td>
</tr>
<tr>
<td>0-&gt;1-&gt;5</td>
<td>6.08</td>
<td>13.38</td>
</tr>
<tr>
<td>0-&gt;1-&gt;3-&gt;5</td>
<td>3.10</td>
<td>13.76</td>
</tr>
<tr>
<td>0-&gt;3-&gt;5</td>
<td>6.99</td>
<td>13.76</td>
</tr>
<tr>
<td>0-&gt;2-&gt;3-&gt;5</td>
<td>8.16</td>
<td>13.76</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>30.01</strong></td>
<td></td>
</tr>
</tbody>
</table>

**Table 5**: Deterministic Approximation Results for Test Network ($\theta=300$)

<table>
<thead>
<tr>
<th>Route</th>
<th>Flow</th>
<th>Perceived Travel Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-&gt;1-&gt;4-&gt;5</td>
<td>4.74</td>
<td>11.26</td>
</tr>
<tr>
<td>0-&gt;1-&gt;5</td>
<td>7.82</td>
<td>11.43</td>
</tr>
<tr>
<td>0-&gt;1-&gt;3-&gt;5</td>
<td>0.90</td>
<td>11.86</td>
</tr>
<tr>
<td>0-&gt;3-&gt;5</td>
<td>7.07</td>
<td>11.55</td>
</tr>
<tr>
<td>0-&gt;2-&gt;3-&gt;5</td>
<td>9.47</td>
<td>11.84</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>30.00</strong></td>
<td></td>
</tr>
</tbody>
</table>
4 Summary and Outlook

In this paper, the stochastic dynamic user optimal (SDUO) route choice model with a link-based formulation is confirmed with the numerical example. The variational inequality approach is shown more general than the other analytical approaches, especially its ability in handling the asymmetric link interactions that does not have a valid optimisation formulation. There are directly solution algorithms to solve a variational inequality formulation. Still, the variational inequality approach is more computationally intensive. The future research will use realistic link travel time functions and implement it on larger network.

Acknowledgements

Danish Transport Council and the Ministry of Transport are thanked for funding the Ph.D. study of the author.

References


