

# A Dynamic Side-Constrained User Equilibrium Problem

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**Abstract:** This paper incorporates two side constraints, namely, the link capacity and first-in-first-out (FIFO) constraints, into the dynamic user equilibrium (DUE) problem. Although the link capacity constraint has been addressed in the literature, its implications for the associated multiple-valued dual variables have not been properly drawn out and, thus, the matter is still subject to controversy. Since the dual variables associated with any side constraint can hardly be realized as true link queuing delays on the physical links, it is more appropriate to regard them as shadow costs (or link toll levies) for avoiding traffic congestion. Interpreting the dual variables as shadow costs (or minimum tolls) reconciles, to a great extent, the arguments that might accrue due to the multiple-valued dual variables. Moreover, this approach implies that the feasible time-space networks used in the first loop operations of the proposed nested diagonalization (ND) method can only be constructed/updated according to the actual link travel times, rather than by the generalized actual link travel times. On the other hand, the FIFO constraint is unusual because it can be activated only when the same physical links in two different time intervals have positive flows. This unusual FIFO constraint is tactically treated by introducing a set of new indicator variables to identify the incidence relationships. The new indicator variables have been incorporated into the equivalence proof between the equilibrium conditions and the variational inequality (VI) model, as well as the proposed ND method. The ND method, embedding the augmented Lagrangian method (ALM), which is in turn coupled with the path-based gradient projection (GP) algorithm, is then demonstrated with a numerical example. The results show that the corresponding dynamic generalized equilibrium conditions are satisfied, since for each origin-destination pair and time interval, the generalized route travel times incurred by the travelers on all used paths are equal and minimal. Although the two side constraints have been incorporated into the DUE problem and their physical meanings have been interpreted in a more plausible way, the interactive effects between these two side constraints are still unknown and, therefore, require more in-depth sensitivity analysis in the future.

**Key Words:** Variational Inequality; Link Capacity Constraint; First-in-first-out Requirement; Dynamic User Equilibrium; Augmented Lagrangian Method; Gradient Projection Algorithm

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## 1 Introduction

Dynamic transportation network models have been intensively studied since intelligent transportation systems (ITS) were first made known to the transportation community. The major difference between the dynamic transportation network models and their static counterparts is the inclusion of a temporal dimension in the former. The addition of a temporal dimension opens up new possibilities for analytical modeling as well as many new research topics, such as traffic flow propagation process, departure time choice, time-window requirements, first-in-first-out (FIFO) conditions, and cyclic phenomenon of intermediate solutions due to time discretization. The literature on these topics consists of numerous published papers [1,9,10,11,21,23] as well as two systematic and comprehensive monographs [5,19].

The two monographs, though having some complementary contents, differ to some extent both in model formulations and solution algorithms. For example, the former employs three link decision variables—link inflow, number of vehicles and exit flow—to describe the flow propagation process, whereas the latter accommodates only one link decision variable—link inflows—in each dynamic network model. While three link decision variables may be more commonly adopted to represent the dynamic link travel time functions, accommodating only one link decision variable can significantly reduce the number of decision variables and, in addition, result in time-dependent network configurations which coincide with time-space diagrams used in the discipline of traffic engineering.

Since a thorough review of the appropriateness of the model formulations and the efficiency of the associated solution algorithms is beyond the scope of this paper, here we only study two specific side constraints, namely, the link capacity and FIFO constraints in connection with the dynamic user equilibrium (DUE) model, leaving other, remaining problems for future research. Link capacity is usually defined as the maximum flow that can pass through a link during an hour. Either a simulation or analytical approach can be adopted to treat link capacity constraint; the latter approach is emphasized in this paper. Tong and Wong [20] formulated a predictive dynamic traffic assignment model and developed a traffic simulator to incrementally load traffic demand onto congested capacity-constrained road networks. The drawback associated with the simulation approach is that intensive computation may be required. In addition, the converged solution may not necessarily coincide with some predetermined travelers' behavior based, for instance, on Wardrop's principles.

Two strategies for dealing with the link capacity constraint can be adopted using analytical models. One is to create a cost function in which the denominator accommodates a term represented by the difference between link capacity and link flow. Whenever the link flow approaches the link capacity, the link travel time function will increase abruptly and so will the value of the objective function in optimization problems, or the value of the gap between the equilibrium solution and the current intermediate solutions in variational inequality (VI) problems. Descent-type solution algorithms embed a mechanism for not allowing such disadvantageous situations to occur during the solution procedure. The difficulty associated with this

strategy is that no such cost function has been justified. The other strategy is to impose the link capacity constraint on the feasible set. Since the link capacity constraint does not fall within the category of basic constraints in standard network equilibrium problems, it is usually regarded as a side constraint. For a variety of reasons, this side-constrained approach is generally preferred [14]. The augmented Lagrangian method (ALM) is commonly adopted to solve the side-constrained problem [18]. Computationally, it is more demanding than the available algorithms for the basic model. Larsson and Patriksson [15] studied a link capacity side-constrained traffic assignment problem and proposed and evaluated an augmented Lagrangean dual method in which the uncapacitated traffic assignment subproblems are solved with a disaggregate simplicial decomposition (DSD) algorithm. The use of dualization approaches for handling the capacity constraints is largely motivated by the efficient algorithms for basic models. Chen and Wang further extended this approach into a dynamic scenario that would better reflect unpredictable situations by including a link capacity constraint expressed in terms of link inflows [7]. However, to solve the uncapacitated user equilibrium subproblems, they replaced the DSD algorithm with the gradient projection (GP) method. Since it is more intuitive in reality, a follow-up study was conducted by Chen, Lee and Chang using an exit flow link capacity constraint instead of an inflow link capacity constraint [3]. For the link inflow capacity model, once the equilibrium solution is obtained via the dualization method, the dynamic extension of Wardrop's first principle (i.e., for each O-D pair and departure time the generalized travel times for used paths are equal and minimal) can also be fully complied with, although it takes more computational time. The dualization approach works well in solving the side-constrained models, but how best to interpret the multiple-valued dual variables associated with the link capacity constraint [14] is a subject for debate. Two interpretations have been proposed: shadow costs (or link toll to be levied) or link queuing delay. Due to the fact that the dual variables are not the true link queuing delay that occurs in reality, the former one, shadow costs (or link toll to be levied), is preferred. With this "virtual cost" interpretation, the arguments caused by the multiple-valued dual variables are largely reconcilable because they are not thought to be as serious as the true link queuing delays. The immediate implication of this interpretation is that the feasible time-space networks in the first loop operations of the proposed nested diagonalization (ND) solution algorithm should only be constructed/updated according to actual link travel times, rather than by generalized link travel times. More detailed discussion about these points can be seen in an earlier version of this paper [4].

The FIFO condition is another notable side constraint which only pertains to the discrete dynamic models. The FIFO requirement, as compared with the link capacity constraint, attracts much less research attention in the transportation community yet remains a critical issue for analytical models. It requires that vehicles can arrive at the destination earlier only by leaving earlier. The occurrence of the first-in-last-out (FILO) phenomenon is generally observed in a situation in which the changing rate of link travel times is markedly high, implying that the continuous-time type models in either a static or dynamic environment should not be of great concern. One may argue that the FILO phenomenon that occurs in the discrete-time models does not necessarily relate to vehicles being overtaken on physical streets and, therefore,

needs no further treatment. However, it is necessary to eliminate the FILO phenomenon in order to avoid possible misunderstanding and further development of the dynamic travel choice models.

Simulation and analytical strategies can be adopted to treat the FIFO requirement. One notable simulation approach derives from the cell transmission model [8], which shows how the evolution of multi-commodity traffic flows over complex networks can be predicted over time based on a simple macroscopic computer representation of traffic flow that is consistent with the kinematic wave theory under all traffic conditions. If the flow propagates according to the cell transmission model then the FIFO condition can be generally observed [17]. The cell transmission model, has to some extent coincided with the concept of a particle-based simulation model [2,12] and has been extensively studied since becoming widely known to the transportation studies community. However, since it contains no explicit cost functions, this simulation approach must be developed in parallel with an analytical modeling approach.

Two approaches to the FIFO requirement can be considered using analytical models. The first approach deals with this issue via the appropriate formulation of the link performance function. The FIFO rule may be satisfied when the link exit function is given as functions of the arrival rate and the queue length [16]. However, this link exit function is hardly accommodated in our analytical model because ours treats link exit flow as variable rather than as a function for interpreting the so-called “linear” flow propagation process, and, moreover, the same vehicle indeed corresponds to the arrival rate, number of vehicles and exit rate on the same link but at different time intervals. The second approach explicitly introduces the FIFO condition into the constraint set. This FIFO constraint can be derived using a simple network with one origin and one destination joined by one link  $a$  for illustration [19]. Suppose the link travel times  $c_a(t)$  and  $c_a(t + \Delta t)$  are generated by vehicles  $u_a(t)$  and  $u_a(t + \Delta t)$  entering link  $a$  during time intervals  $t$  and  $t + \Delta t$ ; the FIFO condition with the analysis time period  $|T|$  may be expressed mathematically as

$$t + c_a(t) \leq t + \Delta t + c_a(t + \Delta t) \quad \forall a, t, \Delta t = 1, 2, 3, \dots, |T| - t, u_a(t) > 0, u_a(t + \Delta t) > 0 \quad (1)$$

Subtracting  $t$  from both sides results in

$$c_a(t) \leq \Delta t + c_a(t + \Delta t) \quad \forall a, t, \Delta t = 1, 2, 3, \dots, |T| - t, u_a(t) > 0, u_a(t + \Delta t) > 0 \quad (2)$$

The above FIFO requirement states for each link  $a$  that the summation of the travel time  $c_a(t + \Delta t)$  and the lag time  $\Delta t$  must be greater than or equal to the travel time  $c_a(t)$ . Note that the FIFO requirement is activated only when the physical link in two different time intervals has positive link flows, i.e.,  $u_a(t) > 0, u_a(t + \Delta t) > 0$ . This special constraint can be tactically

treated by introducing indicator variables into both the equivalence analysis and the solution algorithm.

In fact, the FIFO condition may be expressed in an alternative form by looking backward, that is, in the opposite direction of looking forward. Suppose that for each link  $a$  the link travel times during time intervals  $t$  and  $t - \Delta t$  (rather than intervals  $t$  and  $t + \Delta t$  illustrated by Ran and Boyce [19]) are  $c_a(t)$  and  $c_a(t - \Delta t)$ ; the FIFO condition may be expressed alternatively as

$$t + c_a(t) \geq t - \Delta t + c_a(t - \Delta t) \quad \forall a, t, \Delta t = 1, \dots, t-1, u_a(t) > 0, u_a(t - \Delta t) > 0 \quad (3)$$

Subtracting  $t$  from both sides results in

$$c_a(t) + \Delta t \geq c_a(t - \Delta t) \quad \forall a, t, \Delta t = 1, \dots, t-1, u_a(t) > 0, u_a(t - \Delta t) > 0 \quad (4)$$

The above FIFO requirement states for each link  $a$  that the summation of the travel time  $c_a(t)$  and the lag time  $\Delta t$  must be greater than or equal to the travel time  $c_a(t - \Delta t)$ . Since the above Eqns (2) and (4) are treated similarly in their respective solution procedures and, once equilibrated, result in the same solution [3], only the former one is adopted for demonstration here.

The rest of the paper is organized as follows. In Section 2, the equilibrium conditions corresponding to the DUE problem with the link capacity and FIFO constraints (DUE-CF) are stated and formulated as the VI model. A ND method, embedding the augmented Lagrangian method which, in turn, is coupled with the GP algorithm, is then proposed and elaborated in Section 3. A numerical example is provided for demonstration in Section 4. Finally, concluding remarks are given in Section 5.

## 2 Dynamic User Equilibrium with Link Capacity and First-In-First-Out Constraints

### 2.1 Dynamic Equilibrium Conditions

Assuming O-D demands are fixed and time-dependent, the DUE-CF conditions for each O-D pair and time interval state that the generalized route travel times,  $\widehat{c}_p^{rs*}(k)$  (comprised of travel times,  $c_p^{rs*}(k)$ , and the associate virtual costs,  $\beta_{1p}^{rs*}(k)$  and  $\beta_{2p}^{rs*}(k)$ ), incurred by travelers are equal and minimal, denoted as  $\widehat{\pi}^{rs}(k)$ ; that is, no traveler would be better off by unilaterally changing his/her route. In contrast, the generalized route travel time of any unused route for each O-D pair and time interval is greater than or equal to the minimal generalized route travel times. In other words, at equilibrium, if the flow departing from origin  $r$  during interval  $k$  over route  $p$  toward destination  $s$  is positive, i.e.,  $h_p^{rs*}(k) > 0$ , then the corresponding generalized route travel time is minimal. If, however, no flow occurs on route  $p$ , i.e.,  $h_p^{rs*}(k) = 0$ , then the corresponding generalized route travel time is at least as great as the minimal generalized route travel time. These equilibrium conditions can be mathematically expressed as follows:

$$\widehat{c}_p^{rs*}(k) \begin{cases} = \widehat{\pi}^{rs}(k) & \text{if } h_p^{rs*}(k) > 0 \\ \geq \widehat{\pi}^{rs}(k) & \text{if } h_p^{rs*}(k) = 0 \end{cases} \quad \forall r, s, p, k \quad (5)$$

where

$$\widehat{\pi}^{rs}(k) = \min_p \{ \widehat{c}_p^{rs*}(k) \} \quad \forall r, s, k \quad (6)$$

$$\widehat{c}_p^{rs}(k) = c_p^{rs}(k) + \beta_{1p}^{rs}(k) + \beta_{2p}^{rs}(k) \quad \forall r, s, p, k \quad (7)$$

$$c_p^{rs}(k) = \sum_a \sum_t c_a(t) \delta_{apk}^{rs}(t) \quad \forall r, s, p, k \quad (8)$$

$$\beta_{1p}^{rs}(k) = \sum_a \sum_t \beta_{1a}(t) \delta_{apk}^{rs}(t) \quad \forall r, s, p, k \quad (9)$$

$$\beta_{2p}^{rs}(k) = \sum_a \sum_t \beta_{2a}(t) \delta_{apk}^{rs}(t) \quad \forall r, s, p, k \quad (10)$$

$$\beta_{2a}(t) = \left[ \begin{array}{l} \chi_{2a}(t, t+1)\beta_{2a}(t, t+1) + \chi_{2a}(t, t+2)\beta_{2a}(t, t+2) \\ + \dots + \chi_{2a}(t, |T|)\beta_{2a}(t, |T|) \end{array} \right] c'_a(t) \quad \forall a, t \quad (11)$$

$$c'_a(t) = \frac{\partial c_a(t)}{\partial u_a(t)} \quad \forall a, t \quad (12)$$

Variables of  $\{\bar{\tau}^{rs}(k)\}$ ,  $\{\beta_{1a}(t)\}$ ,  $\{\beta_{2a}(t, t + \Delta t)\}$  are dual variables corresponding, respectively, to the flow conservation constraint (15), link capacity constraint (21) and FIFO constraint (22) below. In this paper, the dual variable associated with the link capacity constraint is termed the link *contemporary* virtual cost, while the dual variable that corresponds to the FIFO constraint is named the *backward* virtual cost. Eqn (5) above states that, optimally, the generalized route travel times on all used paths between any O-D pair during the same interval are equal. The definitional constraints (6)~(11) are self-evident or will become clear later. Eqn (12) defines the partial derivative of cost function with respect to link inflow.

## 2.2 Variational Inequality Formulation

The DUE-CF problem is equivalent to finding a solution  $\mathbf{u}^* \in \Omega_{cf}$  such that the following VI problem holds

$$\sum_a \sum_t c_a^*(t) [u_a(t) - u_a^*(t)] \geq 0 \quad \forall \mathbf{u} \in \Omega_{cf}^* \quad (13)$$

where  $\Omega_{cf}^*$  is an equilibrated subset of the feasible region  $\Omega_{cf}$ , which is delineated by the following constraints.

Flow conservation constraint:

$$\sum_p h_p^{rs}(k) = \bar{q}^{rs}(k) \quad \forall r, s, k \quad (14)$$

Flow propagation constraints:

$$u_{apk}^{rs}(t) = h_p^{rs}(k) \delta_{apk}^{rs}(t) \quad \forall r, s, a, p, k, t \quad (15)$$

$$\sum_t \delta_{apk}^{rs}(t) = 1 \quad \forall r, s, p, a \in p, k \quad (16)$$

$$\delta_{apk}^{rs}(t) = \{0,1\} \quad \forall r, s, a, p, k, t \quad (17)$$

Nonnegativity constraint:

$$h_p^{rs}(k) \geq 0 \quad \forall r, s, p, k \quad (18)$$

Definitional constraints:

$$u_a(t) = \sum_{rs} \sum_p \sum_k h_p^{rs}(k) \delta_{apk}^{rs}(t) \quad \forall a, t \quad (19)$$

$$c_p^{rs}(k) = \sum_a \sum_t c_a(t) \delta_{apk}^{rs}(t) \quad \forall r, s, p, k \quad (20)$$

Link capacity constraint:

$$u_a(t) \leq CAP_a(t) \quad \forall a, t \quad (21)$$

First-in-first-out constraint:

$$c_a(t) \leq \Delta t + c_a(t + \Delta t) \quad \forall a, t, \Delta t = 1, 2, 3, \dots, |T| - t, u_a(t) > 0, u_a(t + \Delta t) > 0 \quad (22)$$

Eqn (14) conserves the route flows. Eqn (15) describes the flow propagation through the use of the indicator variables. If the indicator variable  $\delta_{apk}^{rs}(t)$  is equal to 1, then the route flow  $h_p^{rs}(k)$  will be realized as  $u_{apk}^{rs}(t)$ . In contrast, if the indicator variable  $\delta_{apk}^{rs}(t)$  is equal to zero, it implies that route flow  $h_p^{rs}(k)$  does not constitute link flow  $u_a(t)$ . Eqn (16) indicates that time-dependent link  $(a,t)$  can be incident to time-dependent route  $(p,k)$  once, at most. If the route flow is not presented on link  $a$ , then it must be put on one of the other links in the network, unless the destination has been reached. Eqn (17) designates that indicator variables are integer-valued, implying that flow deformation is not possible in our model. Eqn (18) ensures that all route flows are nonnegative. Eqn (19) expresses the link flows in terms of the route flows through the use of the indicator variables. Eqn (20) expresses the route travel time by the linear combination of all link travel times along that route. Eqn (21) ensures that link in-

flow must be less than or equal to link capacity. Eqn (22) requires the FIFO condition be preserved. Note that the FIFO condition is activated only when inflows on link  $a$  at time intervals  $t$  and  $t + \Delta t$  are greater than zero, that is,  $u_a(t) > 0$  and  $u_a(t + \Delta t) > 0$ . In other words, when one or two of two different time intervals associated with the same physical link have nil flow, the FIFO requirement is redundant. (The equivalence between the equilibrium conditions (5) and the VI model (13) is addressed in Appendix I.)

### 3 The Nested Diagonalization Method

For solving the DUE-CF problem, it is necessary to develop a nested diagonalization (ND) method. The basic idea behind the ND method is to repetitively relax all asymmetric link interactions and then solve the resulting side-constrained optimization submodel. Two types of asymmetric link interactions have been identified: actual link travel times and the inflows other than on the subject time-space link. Once these two types of asymmetric link interactions are temporarily fixed, the resulting side-constrained optimization subproblem can then be solved by the augmented Lagrangian method which essentially dualizes the two side constraints to form a standard user equilibrium problem to be solved by the GP method. The proposed ND method essentially contains four loops: an outermost loop in which the actual link travel times are estimated; a second loop in which all the inflows other than the subject time-space link are temporarily fixed; a third loop in which the dual variables are estimated; and an innermost loop in which the standard traffic assignment problem is solved.

Keeping in mind the discussion thus far, the ND method is formally proposed as follows.

#### *The ND Method*

##### **Step 0: Initialization.**

Step 0.1: *First Loop Operation.*

Let  $m=0$ . Set  $\{\tau_a^m(t)\} = \{NINT[c_{a_0}(t)]\}$ . Construct the corresponding time-space network.

Step 0.2: *Second Loop Operation.*

Let  $n=0$ . Set  $\{c_a^n(t)\} = \{c_{a_0}(t)\}$ . Fix all the inflows at the current level except for the subject time-space link.

Step 0.3: *Third Loop Operation.*

Let  $l=0$ . Set initial Lagrangian multipliers  $\{\beta_{1a}^l(t)\} = \{0\}$  and  $\beta_{2a}^l(t, t + \Delta t) = 0, \forall a, t, \Delta t = 1, 2, 3, \dots, |T| - t$ .

Step 0.4: *Fourth Loop Operation.*

Solve for the solution  $\{h_p^{rs}(k)\}$  and the corresponding  $\{u_a^l(t)\}$  in the following optimization problem using the GP method.

$$\begin{aligned} \max_{\beta \geq 0} \min_{\mathbf{h} \in \bar{\Omega}} L(\mathbf{h}, \beta) = & \sum_a \sum_t \int_0^{u_a(t)} c_a(\mathbf{u} \setminus u_a(t), \omega) d\omega + \sum_a \sum_t \beta_{1a}(t) [u_a(t) - CAP_a(t)] \\ & + \sum_a \sum_t \left\{ \begin{aligned} & \chi_{2a}(t, t+1) \beta_{2a}(t, t+1) [c_a(t) - 1 - c_a(t+1)] \\ & + \chi_{2a}(t, t+2) \beta_{2a}(t, t+2) [c_a(t) - 2 - c_a(t+2)] \\ & + \dots \\ & + \chi_{2a}(t, |T|) \beta_{2a}(t, |T|) [c_a(t) - (|T| - t) - c_a(|T|)] \end{aligned} \right\} \end{aligned} \quad (23)$$

Where the feasible region  $\bar{\Omega}$  is bounded by a subset of feasible region (14)~(20) with a certain propagation relationship  $\{\delta_{apk}^{rs}(t)\} = \{\bar{\delta}_{apk}^{rs}(t)\}$ .

Step 0.5: *Convergence Check for the Third Loop Operation.*

If  $\max_{a,t} \{u_a^l(t) - CAP_a(t)\} \leq 0.0001$ ,

$\max_{a,t,\Delta t=1,2,\dots,|T|-t} \{\chi_{2a}(t, t+\Delta t) [c_a^l(t) - c_a^l(t+\Delta t) - \Delta t]\} \leq 0.0001$  and

$\max_{a,t,\Delta t=1,2,\dots,|T|-t} \{\beta_{2a}^l(t, t+\Delta t) [c_a^l(t) - c_a^l(t+\Delta t) - \Delta t]\} \leq 0.001$ ,

let  $\{c_a^{n+1}(t)\} = \{c_a^l(t)\}$ ,  $\{\beta_{1a}^{n+1}(t)\} = \{\beta_{1a}^l(t)\}$  and  $\{\beta_{2a}^{n+1}(t, t+\Delta t)\} = \{\beta_{2a}^l(t, t+\Delta t)\}$ ,

go to Step 0.6. Otherwise, update the Lagrangian multipliers  $\{\beta_{1a}^{l+1}(t)\}$  and  $\{\beta_{2a}^{l+1}(t, t+\Delta t)\}$  using the following formula. Set  $l = l+1$  and go to Step 0.4.

$$\beta_{1a}^{l+1}(t) = \begin{cases} \beta_{1a}^l(t) + \xi_1 [c_a^l(u) - c_a^l(\mathbf{u} \setminus u_a(t), CAP_a(t))] & \text{if } u_a^l(t) > Cap_a(t) \\ \frac{1}{2} \beta_{1a}^l(t) & \text{if } u_a^l(t) \leq Cap_a(t) \end{cases} \quad (24)$$

$$\beta_{2a}^{l+1}(t, t + \Delta t) = \begin{cases} \frac{\beta_{2a}^l(t, t + \Delta t)c'_a(t)|_{u_a^l(t)} + \xi_2 [c_a^l(u) - c_a^l(\mathbf{u} \setminus u_a(t), u_a(t + \Delta t)) + \Delta t]}{c'_a(t)|_{u_a^l(t)}}, & \text{if } \chi_{2a}(t, t + \Delta t) = 1 \text{ and } c_a^l(t + \Delta t) + \Delta t < c_a^l(t) \\ \frac{1}{2} \beta_{2a}^l(t, t + \Delta t) & \text{if } \chi_{2a}(t, t + \Delta t) = 1 \text{ and } c_a^l(t + \Delta t) + \Delta t \geq c_a^l(t), \\ & \text{or } \chi_{2a}(t, t + \Delta t) = 0 \end{cases} \quad (25)$$

where  $\xi_1$  and  $\xi_2$  are step sizes for Lagrangian multipliers  $\{\beta_{1a}^{l+1}(t)\}$  and  $\{\beta_{2a}^{l+1}(t, t + \Delta t)\}$ .  $\xi_1$  and  $\xi_2$  are set to 1.

Step 0.6: Convergence Check for the Second Loop Operation.

If  $\max_{a,t} \left| \frac{u_a^{n+1}(t) - u_a^n(t)}{u_a^{n+1}(t)} \right| \leq 0.0001$ , go to Step 1. Otherwise, set  $n=n+1$ , go to Step 0.3.

**Step 1: Subproblem Solving Operation.**

Step 1.1: First Loop Operation.

Let  $m = m+1$ . Update the estimated actual link travel times by

$$\tau_a^m(t) = NINT \{ (1 - \gamma)\tau_a^{m-1}(t) + \gamma c_a^{n+1}(t) \} \quad \forall a, t \quad (26)$$

where  $0 < \gamma \leq 1$  and symbol  $NINT\{\cdot\}$  denotes the integer arithmetic operator. Construct the corresponding feasible time-space network based on the estimated actual link travel times  $\{\tau_a^m(t)\}$ .

Step 1.2: Second Loop Operation.

Let  $n=0$ . Set  $\{c_a^n(t)\} = \{c_{a_0}(t)\}$ . Fix all the inflows at the current level except for the subject time-space link.

Step 1.3: Third Loop Operation.

Let  $l=0$ . Set initial Lagrangian multipliers  $\{\beta_{1a}^l(t)\} = \{0\}$  and  $\{\beta_{2a}^l(t, t + \Delta t)\} = \{0\}$ .

Step 1.4: Fourth Loop Operation.

Solve for the solution  $\{h_p^{rs}(k)\}$  and the corresponding  $\{u_a^l(t)\}$  in the optimization problem (23) using the GP method.

Step 1.5: Convergence Check for the *Third Loop Operation*.

If  $\max_{a,t} \{u_a^l(t) - CAP_a(t)\} \leq 0.0001$ ,

$\max_{a,t,\Delta t=1,2,\dots,|T|-t} \{\chi_{2a}(t, t + \Delta t)[c_a^l(t) - c_a^l(t + \Delta t) - \Delta t]\} \leq 0.0001$

and  $\max_{a,t,\Delta t=1,2,\dots,|T|-t} \{\beta_{2a}^l(t, t + \Delta t)[c_a^l(t) - c_a^l(t + \Delta t) - \Delta t]\} \leq 0.001$ , let

$\{c_a^{n+1}(t)\} = \{c_a^l(t)\}$ ,  $\{\beta_{1a}^{n+1}(t)\} = \{\beta_{1a}^l(t)\}$  and  $\{\beta_{2a}^{n+1}(t, t + \Delta t)\} = \{\beta_{2a}^l(t, t + \Delta t)\}$ , go to

Step 1.6. Otherwise, update the Lagrangian multipliers  $\{\beta_{1a}^{l+1}(t)\}$  and  $\{\beta_{2a}^{l+1}(t, t + \Delta t)\}$  using Eqns (24)~(25). Let  $l=l+1$  and go to Step 1.4.

Step 1.6: Convergence Check for the *Second Loop Operation*.

If  $\max_{a,t} \left| \frac{u_a^{n+1}(t) - u_a^n(t)}{u_a^{n+1}(t)} \right| \leq 0.0001$ , go to Step 2; otherwise, set  $n=n+1$ , go to Step 1.3.

**Step 2: Convergence Check for the First Loop Operation.**

If  $\tau_a^m(t) = NINT[c_a^{n+1}(t)]$ ,  $\forall a, t$ , stop; the current solution is optimal. Otherwise, set  $n=n+1$ , go to Step 1.

Steps 0.3~0.5 and 1.3~1.5 are essentially the augmented Lagrangian method. The time-dependent traffic assignment problem in Steps 0.4 and 1.4 is solved by the GP method. A detailed description of the GP method is described in Appendix III. Note that in Steps 0.1 and 1.1, the actual link travel times are updated by the real link travel times, rather than the generalized link travel times. Note also that the indicator variables have been used in the construction of the Lagrangian function in Steps 0.4 and 1.4, and the convergence check in Steps 0.5 and 1.5.

## 4 Numerical Example

### 4.1 Data

The simple network shown in Figure 1 is used for testing. The test network consists of 6 links and 5 nodes.

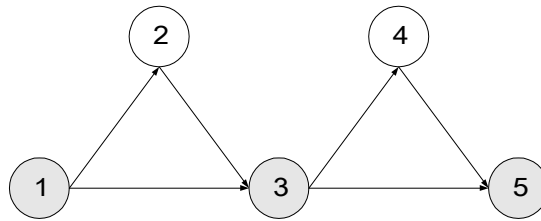


Figure 1: Test Network

The dynamic link travel time function is arbitrarily constructed as follows (because its practical form has yet to be worked out):

$$c_a(t) = 1 + 0.01(u_a(t))^2 + 0.001(x_a(t))^2 \quad \forall a, t \tag{27}$$

where  $u_a(t)$  denotes the inflows on link  $a$  during interval  $t$ , and  $x_a(t)$  indicates the number of vehicles on link  $a$  at the beginning of interval  $t$ . Therefore,

$$c'_a(t) = \frac{\partial c_a(t)}{\partial u_a(t)} = 0.02u_a(t) \quad \forall a, t \tag{28}$$

The time-dependent origin-destination (O-D) demands are assumed as time-independent in Table 1.

Table 1: Time-Dependent O-D Demands

| O-D Pair | Time Interval | Demand |
|----------|---------------|--------|
| 1-5      | 1             | 15.00  |
|          | 2             | 13.00  |
| 3-5      | 1             | 20.00  |
|          | 2             | 5.00   |

The capacity constraint is only imposed on link 1→3, which allows no link inflow larger than 8 units.

### 4.2 Test Results

Both the DUE and DUE-CF models were solved. The corresponding link-related results are summarized in Tables 2 and 3.

Table 2 summarizes the result without the link capacity and FIFO constraints. Unfortunately, the link capacity constraint is violated on link 1 → 3 during intervals 1 and 2, because the associated link inflows are 11.29 and 10.25 units which obviously exceed the magnitude of 8.00 units. As for the FIFO requirement, violation can also be observed by comparing the wall clock times for the flow exiting link 3 → 5 during intervals 1 and 2. The associated wall clock times can be computed below:

$$1 + c_{3 \rightarrow 5}(1) = 1 + 2.84 = 3.84 \geq 3.43 = 2 + 1.43 = 2 + c_{3 \rightarrow 5}(4) \tag{29}$$

**Table 2:** Link Results for the DUE Model  
(Without Link Capacity and FIFO Constraints)

| Link  | Entering Time Interval | Link Cap. | Inflow       | Exit Flow | Number of Vehicles | Link Travel Time | Contemp. Virtual cost $\beta_{1a}(t)$ | Backward Virtual cost $\beta_{2a}(t)$                        | Gener. Link Travel Time | Exiting Time Interval |
|-------|------------------------|-----------|--------------|-----------|--------------------|------------------|---------------------------------------|--|-------------------------|-----------------------|
|       |                        |           |              |           |                    |                  |                                       | $(t, t + \Delta t):$<br>$\beta_{2a}(t, t + \Delta t)c'_a(t)$ |                         |                       |
| 1 → 2 | 1                      | --        | 3.71         | 0.00      | 0.00               | 1.14             | 0.00                                  | 0.00   | 1.14                    | 2                     |
|       | 2                      | --        | 2.75         | 3.71      | 3.71               | 1.09             | 0.00                                  | 0.00   | 1.09                    | 3                     |
|       | 3                      | --        | 0.00         | 2.75      | 2.75               | 1.01             | 0.00                                  | 0.00   | 1.01                    | -                     |
| 1 → 3 | <b>1</b>               | --        | <b>11.29</b> | 0.00      | 0.00               | 2.28             | 0.00                                  | 0.00   | 2.28                    | 3                     |
|       | <b>2</b>               | --        | <b>10.25</b> | 0.00      | 11.29              | 2.18             | 0.00                                  | 0.00   | 2.18                    | 4                     |
|       | 3                      | --        | 0.00         | 11.29     | 21.54              | 1.46             | 0.00                                  | 0.00   | 1.46                    | -                     |
|       | 4                      | --        | 0.00         | 10.25     | 10.25              | 1.11             | 0.00                                  | 0.00   | 1.11                    | -                     |
| 2 → 3 | 2                      | --        | 3.71         | 0.00      | 0.00               | 1.14             | 0.00                                  | 0.00   | 1.14                    | 3                     |
|       | 3                      | --        | 2.75         | 3.71      | 3.71               | 1.09             | 0.00                                  | 0.00   | 1.09                    | 4                     |
|       | 4                      | --        | 0.00         | 2.75      | 2.75               | 1.01             | 0.00                                  | 0.00   | 1.01                    | -                     |
| 3 → 4 | 1                      | --        | 6.45         | 0.00      | 0.00               | 1.42             | 0.00                                  | 0.00   | 1.42                    | 2                     |
|       | 2                      | --        | 0.00         | 6.45      | 6.45               | 1.04             | 0.00                                  | 0.00   | 1.04                    | -                     |
|       | 3                      | --        | 4.61         | 0.00      | 0.00               | 1.21             | 0.00                                  | 0.00   | 1.21                    | 4                     |
|       | 4                      | --        | 4.06         | 4.61      | 4.61               | 1.19             | 0.00                                  | 0.00   | 1.19                    | 5                     |
|       | 5                      | --        | 0.00         | 4.06      | 4.06               | 1.02             | 0.00                                  | 0.00   | 1.02                    | -                     |
| 3 → 5 | <b>1</b>               | --        | 13.55        | 0.00      | 0.00               | <b>2.84</b>      | 0.00                                  | 0.00   | <b>2.84</b>             | <b>4</b>              |
|       | <b>2</b>               | --        | 5            | 0.00      | 13.55              | <b>1.43</b>      | 0.00                                  | 0.00   | <b>1.43</b>             | <b>3</b>              |
|       | 3                      | --        | 10.39        | 5         | 18.55              | 2.42             | 0.00                                  | 0.00   | 2.42                    | 5                     |
|       | 4                      | --        | 8.94         | 13.55     | 23.94              | 2.37             | 0.00                                  | 0.00   | 2.37                    | 6                     |
|       | 5                      | --        | 0.00         | 10.39     | 19.33              | 1.37             | 0.00                                  | 0.00   | 1.37                    | -                     |
|       | 6                      | --        | 0.00         | 8.94      | 8.94               | 1.08             | 0.00                                  | 0.00   | 1.08                    | -                     |

|       |   |    |      |      |      |      |      |      |      |   |
|-------|---|----|------|------|------|------|------|------|------|---|
| 4 → 5 | 2 | -- | 6.45 | 0.00 | 0.00 | 1.42 | 0.00 | 0.00 | 1.42 | 3 |
|       | 3 |    | 0.00 | 6.45 | 6.45 | 1.04 | 0.00 | 0.00 | 1.04 | - |
|       | 4 | -- | 4.61 | 0.00 | 0.00 | 1.21 | 0.00 | 0.00 | 1.21 | 5 |
|       | 5 | -- | 4.06 | 4.61 | 4.61 | 1.19 | 0.00 | 0.00 | 1.19 | 6 |
|       | 6 | -- | 0.00 | 4.06 | 4.06 | 1.02 | 0.00 | 0.00 | 1.02 | - |

Table 3 shows that the violation of the link capacity and FIFO constraints has been resolved by explicitly adding these constraints into the original model. On link 1→3 during time interval 1 and 2, the inflows were reduced to 8.00 which complies with the capacity constraint. To show that the FIFO constraints are preserved, the wall clock times for the flow exiting link 3→5 during intervals 1 and 2 are calculated for comparison as follows:

$$1 + c_{3 \rightarrow 5}(1) = 1 + 2.39 = 3.39 \leq 3.39 = 2 + 1.39 = 2 + c_{3 \rightarrow 5}(5) \tag{30}$$

**Table 3:** Link Results for the DUE-CF Model

| Link  | Entering Time Interval | Link Cap. | Inflow      | Exit Flow | Number of Vehicles | Link Travel Time | Contemp. Virtual Cost $\beta_{1a}(t)$ | Backward Virtual Cost $\beta_{2a}(t)$                   | Gener. Link Travel Time | Exiting Time Interval |
|-------|------------------------|-----------|-------------|-----------|--------------------|------------------|---------------------------------------|---|-------------------------|-----------------------|
|       |                        |           |             |           |                    |                  |                                       | $(t, t + \Delta t): \beta_{2a}(t, t + \Delta t)c'_a(t)$ |                         |                       |
| 1 → 2 | 1                      | --        | 7.00        | 0.00      | 0.00               | 1.49             | 0.00                                  | 0.00  | 1.49                    | 2                     |
|       | 2                      | --        | 5.00        | 7.00      | 7.00               | 1.30             | 0.00                                  | 0.00  | 1.30                    | 3                     |
|       | 3                      | --        | 0.00        | 5.00      | 5.00               | 1.03             | 0.00                                  | 0.00  | 1.03                    | -                     |
| 1 → 3 | <b>1</b>               | 8.00      | <b>8.00</b> | 0.00      | 0.00               | 1.64             | 1.34                                  | 0.00  | 2.98                    | 3                     |
|       | <b>2</b>               | 8.00      | <b>8.00</b> | 0.00      | 8.00               | 1.70             | 0.89                                  | 0.00  | 2.60                    | 4                     |
|       | 3                      | 8.00      | 0.00        | 8.00      | 16.00              | 1.26             | 0.00                                  | 0.00  | 1.26                    | -                     |
|       | 4                      | 8.00      | 0.00        | 8.00      | 8.00               | 1.06             | 0.00                                  | 0.00  | 1.06                    | -                     |
| 2 → 3 | 2                      | --        | 7.00        | 0.00      | 0.00               | 1.49             | 0.00                                  | 0.00  | 1.49                    | 3                     |
|       | 3                      | --        | 5.00        | 7.00      | 7.00               | 1.30             | 0.00                                  | 0.00  | 1.30                    | 4                     |
|       | 4                      | --        | 0.00        | 5.00      | 5.00               | 1.02             | 0.00                                  | 0.00  | 1.03                    | -                     |
| 3 → 4 | 1                      | --        | 8.21        | 0.00      | 0.00               | 1.67             | 0.00                                  | 0.00  | 1.67                    | 3                     |
|       | 2                      | --        | 0.00        | 0.00      | 8.21               | 1.07             | 0.00                                  | 0.00  | 1.07                    | -                     |
|       | 3                      | --        | 4.10        | 8.21      | 8.21               | 1.24             | 0.00                                  | 0.00  | 1.24                    | 4                     |
|       | 4                      | --        | 2.27        | 4.10      | 4.10               | 1.07             | 0.00                                  | 0.00  | 1.07                    | 5                     |
|       | 5                      | --        | 0.00        | 2.27      | 2.27               | 1.01             | 0.00                                  | 0.00  | 1.01                    | -                     |
| 3 → 5 | <b>1</b>               | --        | 11.80       | 0.00      | 0.00               | <b>2.39</b>      | 0.00                                  | <b>0.96</b>   | <b>3.35</b>             | <b>3</b>              |
|       | <b>2</b>               | --        | 5.00        | 0.00      | 11.79              | <b>1.39</b>      | 0.00                                  | <b>0.00</b>   | <b>1.39</b>             | <b>3</b>              |
|       | 3                      | --        | 10.90       | 16.79     | 16.79              | 2.47             | 0.00                                  | 0.00  | 2.47                    | 5                     |
|       | 4                      | --        | 10.73       | 0.00      | 10.90              | 2.27             | 0.00                                  | 0.00  | 2.27                    | 6                     |
|       | 5                      | --        | 0.00        | 10.90     | 21.64              | 1.47             | 0.00                                  | 0.00  | 1.47                    | -                     |
|       | 6                      | --        | 0.00        | 10.73     | 10.73              | 1.12             | 0.00                                  | 0.00  | 1.12                    | -                     |
| 4 → 5 | 3                      | --        | 8.21        | 0.00      | 0.00               | 1.67             | 0.00                                  | 0.00  | 1.67                    | 5                     |
|       | 4                      | --        | 4.10        | 0.00      | 8.21               | 1.24             | 0.00                                  | 0.00  | 1.24                    | 5                     |
|       | 5                      | --        | 2.27        | 12.30     | 12.30              | 1.20             | 0.00                                  | 0.00  | 1.20                    | 6                     |
|       | 6                      | --        | 0.00        | 2.27      | 2.27               | 1.01             | 0.00                                  | 0.00  | 1.01                    | -                     |

The rationale behind the proposed model and associated solution algorithm can be verified by checking if the resulting generalized route travel times satisfy the equilibrium conditions defined in Section 3. The generalized route travel times corresponding to Tables 2 and 3 are summarized respectively in Tables 4 and 5. Consider route  $1 \rightarrow 3 \rightarrow 5$  departing from origin 1 during interval 1; the respective generalized route travel times for Tables 4 and 5 can be obtained by summing up the route cruising travel time, contemporary virtual cost and backward virtual cost as follows.

$$\begin{aligned} \hat{c}_{1 \rightarrow 3 \rightarrow 5}(1) &= c_{1 \rightarrow 3 \rightarrow 5}(1) + (\beta_{1(1 \rightarrow 3)}(1) + \beta_{1(3 \rightarrow 5)}(3)) + (\beta_{2(1 \rightarrow 3)}(1) + \beta_{2(3 \rightarrow 5)}(3)) \\ &= 4.70 + (0.00 + 0.00) + (0.00 + 0.000) = 4.70 \end{aligned} \tag{31}$$

$$\begin{aligned} \hat{c}_{1 \rightarrow 3 \rightarrow 5}(1) &= c_{1 \rightarrow 3 \rightarrow 5}(1) + (\beta_{1(1 \rightarrow 3)}(1) + \beta_{1(3 \rightarrow 5)}(3)) + (\beta_{2(1 \rightarrow 3)}(1) + \beta_{2(3 \rightarrow 5)}(3)) \\ &= 4.11 + (1.34 + 0.00) + (0.00 + 0.000) = 5.45 \end{aligned} \tag{32}$$

The remaining generalized route travel times are also computed and summarized in Tables 4 and 5. As can be observed, the travelers departing from the same origin during the same interval also experience approximately the same generalized route travel time.

**Table 4:** Generalized Route Travel Times for the DUE Model  
(Without Link Capacity and FIFO Constraints)

| O-D Pair | Entering Time Interval | Route             | Inflow | Route Travel Time | Contemp. Virtual Cost | Backward Virtual Cost | Generalized Route Travel Time |
|----------|------------------------|-------------------|--------|-------------------|-----------------------|-----------------------|-------------------------------|
| 1 → 5    | 1                      | 1 → 3 → 5         | 9.88   | 4.70              | 0.00                  | 0.00                  | 4.70                          |
|          |                        | 1 → 2 → 3 → 4 → 5 | 3.20   | 4.70              | 0.00                  | 0.00                  | 4.70                          |
|          |                        | 1 → 2 → 3 → 5     | 0.51   | 4.70              | 0.00                  | 0.00                  | 4.70                          |
|          |                        | 1 → 3 → 4 → 5     | 1.41   | 4.70              | 0.00                  | 0.00                  | 4.70                          |
|          | 2                      | 1 → 3 → 5         | 8.94   | 4.55              | 0.00                  | 0.00                  | 4.55                          |
|          |                        | 1 → 3 → 4 → 5     | 1.31   | 4.55              | 0.00                  | 0.00                  | 4.55                          |
|          |                        | 1 → 2 → 3 → 4 → 5 | 2.75   | 4.55              | 0.00                  | 0.00                  | 4.55                          |
| 3 → 5    | 1                      | 3 → 5             | 13.55  | 2.84              | 0.00                  | 0.00                  | 2.84                          |
|          |                        | 3 → 4 → 5         | 6.45   | 2.83              | 0.00                  | 0.00                  | 2.83                          |
|          | 2                      | 3 → 5             | 5.00   | 1.43              | 0.00                  | 0.00                  | 1.43                          |
| Total    |                        |                   |        | 193.60            | 0                     | 0                     | 193.60                        |

**Table 5:** Generalized Route Travel Times for the Test Network  
(With Link Capacity and FIFO Constraints)

| O-D Pair | Entering Time Interval | Route             | Inflow | Route Travel Time | Contemp. Virtual Cost | Backward Virtual Cost | Generalized Route Travel Time |
|----------|------------------------|-------------------|--------|-------------------|-----------------------|-----------------------|-------------------------------|
| 1 → 5    | 1                      | 1 → 3 → 5         | 8.00   | 4.11              | 1.34                  | 0.00                  | 5.45                          |
|          |                        | 1 → 2 → 3 → 4 → 5 | 4.10   | 5.45              | 0.00                  | 0.00                  | 5.45                          |
|          |                        | 1 → 2 → 3 → 5     | 2.90   | 5.45              | 0.00                  | 0.00                  | 5.45                          |
|          | 2                      | 1 → 3 → 5         | 8.00   | 3.97              | 0.89                  | 0.00                  | 4.87                          |
|          |                        | 1 → 2 → 3 → 5     | 2.73   | 4.87              | 0.00                  | 0.00                  | 4.87                          |
|          |                        | 1 → 2 → 3 → 4 → 5 | 2.27   | 4.87              | 0.00                  | 0.00                  | 4.87                          |
| 3 → 5    | 1                      | 3 → 5             | 11.79  | 2.39              | 0.00                  | 0.96                  | 3.35                          |
|          |                        | 3 → 4 → 5         | 8.21   | 3.35              | 0.00                  | 0.00                  | 3.35                          |
|          | 2                      | 3 → 5             | 5.00   | 1.39              | 0.00                  | 0.00                  | 1.39                          |
| Total    |                        |                   |        | 187.82            | 17.84                 | 11.32                 | 219.01                        |

For each O-D pair, the route travel times on used paths are the same for the DUE model but are not necessarily the same for the DUE-CF model. However, the generalized route travel times (including the route travel times and the corresponding virtual costs) are identical and comply with the equilibrium conditions shown in Eqn (5). Note that the total network travel time for the DUE model is 193.60 units, which is higher than the 187.82 units associated with the DUE-CF model. This is simply because the gains from pushing flows out of the links that violate link capacity and FIFO constraints are greater than the loss incurred due to the increase of flows on the other links. However, the total generalized network travel time for the DUE model is 193.60 units, which is less than the 219.01 units associated with the DUE-CF model. This result is intuitive because, in terms of generalized network travel time, the more restrictive the feasible region, the higher the objective value.

## 5 CONCLUSION AND SUGGESTIONS

In this paper, the link capacity and FIFO constraints were incorporated into the DUE problem. The capacity constraint has been studied for years in both static and dynamic cases [7,15,22]. However, the multiple-valued dual variables associated with the link capacity constraint,  $\{\beta_{1a}(t)\}$ , have not been appropriately interpreted up to now. Due to the fact that the dual variables  $\{\beta_{1a}(t)\}$  are not really related to link queuing delays, it would be better to interpret them as “contemporary virtual link costs” (or as tolls to be levied) to prevent unfavorable traffic congestion. When this “virtual cost” interpretation is adopted, arguments about the multiple-valued dual variable are significantly diminished because no true link queuing delays really come into play. In addition, an immediate implication, in this case, is that only the link travel

times, rather than the generalized link travel times, should be used in the first loop operations of the proposed ND method for constructing and updating the feasible time-space networks.

The second side constraint, that is, the FIFO requirement, has received less study up to now and no meaningful numerical examples have been provided in the literature. The FIFO constraint can only be activated when the same physical links in the two different time intervals have positive flows. This side constraint is unusual and indeed requires special treatment. To manage it, a new set of indicator variables is introduced to help identify incidence relationships. This tactic, though not seen before, works well in our case. With this new set of indicator variables, the equivalence of the equilibrium conditions and VI model formulation can be easily proved. Moreover, within the ND procedure, the Lagrangian function can be constructed and the convergence criterion can be readily checked. Similar to the link capacity constraint, the variables associated with the FIFO requirement  $\{\beta_{2a}(t, t + \Delta t)\}$  are termed “backward virtual link costs” for ease of reference.

The numerical example shows that the two side constraints can be effectively satisfied by the proposed ND method. In addition, having defined the general link travel time (comprised of three components—link travel time, contemporary and backward virtual link costs), the results show that the corresponding DUE-CF conditions (the minimum and equal generalized route travel time for each O-D pair and time interval) are fully compliant with it.

Although the link capacity and FIFO constraints have been covered thoroughly in this paper, the interactive effects between these two side constraints is still unknown. For instance, does the contemporary virtual link cost constitute a component of the backward virtual link cost? If so, in what form? More in-depth sensitivity analysis is thus required. Note also that, up to this point, the two most crucial unsolved issues remain for the dynamic travel choice problems: the correct forms of dynamic travel time functions and the nonconvergence (or cyclic) phenomenon. More effort on these two issues is urgently needed to expedite the full-scale application of dynamic travel choice models, which will certainly be of great help in the future advancement of intelligence transportation systems.

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## APPENDIX I: Notation

The following is a summary of the symbols used in this paper.

$a$  link designation

$(a,t)$  time-dependent link designation

$A = \{(a,t) \mid (a,t) \in \{(\bar{\delta}_{apk}^{rs}(t)=1) \wedge (\bar{\delta}_{ap\hat{k}}^{rs}(t)=0)\}\}$ ; set of time-dependent links in time-dependent non-shortest path  $(p,k)$ , but not in time-dependent shortest path  $(\hat{p},\hat{k})$

- $\hat{A}$   $\hat{A} = \{(a, t) \mid (a, t) \in \{(\bar{\delta}_{apk}^{rs}(t) = 0) \wedge (\bar{\delta}_{apk}^{rs}(t) = 1)\}\}$ ; set of time-dependent links not in time-dependent non-shortest path  $(p, k)$ , but in time-dependent shortest path  $(\hat{p}, \hat{k})$
- $c_a(t)$  travel time for link  $a$  during time interval  $t$
- $c_{a_0}(t)$  free flow travel time for link  $a$  during interval  $t$
- $\bar{c}_a(t)$  generalized travel time for link  $a$  during time interval  $t$
- $c'_a(t)|_{u_a^l(t)}$  derivative of link travel time function with respect to inflow that is evaluated at  $(\bar{\mathbf{u}} \setminus \bar{u}_a(t), u_a(t))$
- $c_p^{rs}(k)$  travel time for route  $p$  between O-D pair  $rs$  during time interval  $k$
- $\bar{c}_p^{rs}(k)$  generalized travel time for route  $p$  between O-D pair  $rs$  during time interval  $k$
- $CAP_a(t)$  capacity for link  $a$  during time interval  $t$
- $d_p^{rs}(k)$  descent direction for route  $p$  between O-D pair  $rs$  during time interval  $k$
- $\mathbf{h}$  vector of route flow
- $h_p^{rs}(k)$  flow on route  $p$  between O-D pair  $rs$  during time interval  $k$
- $j$  node designation
- $k$  time interval designation (which is usually associated with a route)
- $p$  route designation
- $(p, k)$  time-dependent route designation
- $(\hat{p}, \hat{k})$  shortest time-dependent route designation
- $\bar{q}^{rs}(k)$  fixed traffic demand between O-D pair  $rs$  during time interval  $k$
- $r$  origin designation

- $s$  destination designation
- $t$  time interval designation (which is usually associated with a link)
- $|T|$  total number of time intervals
- $\mathbf{u}$  vector of link inflows
- $(\bar{\mathbf{u}} \setminus \bar{u}_a(t), u_a(t))$  vector of inflows being fixed at current level except inflow on link  $a$  during time interval  $t$
- $u_a(t)$  inflow on link  $a$  during time interval  $t$
- $x_a(t)$  number of vehicles on link  $a$  at the beginning of time interval  $t$
- $z(\mathbf{u})$  objective function
- $\alpha$  move size
- $\beta_a(t)$  total virtual cost associated with link  $a$  during interval  $t$
- $\beta_{1a}(t)$  dual variable associated with link capacity constraint for link  $a$  during time interval  $t$ ; also defined as contemporary virtual cost for link  $a$  during time interval  $t$
- $\beta_{2a}(t)$  summation of dual variable  $\beta_{2a}(t, t + \Delta t)$  over all lead time  $\Delta t$ , i.e.,  

$$\beta_{2a}(t) = \sum_{\Delta t=1}^{|T|-t} \beta_{2a}(t, t + \Delta t) c'_a(t), \forall a, t$$
; also defined as backward virtual cost for link  $a$  during time interval  $t$ ;
- $\beta_{2a}(t, t + \Delta t)$  dual variable associated with the first-in-first-out constraint for link  $a$  between time intervals  $t$  and  $(t + \Delta t)$ , where  $\Delta t = 1, \dots, |T| - t$
- $\beta_p^{rs}(k)$  virtual cost associated with route  $p$  between O-D pair  $rs$  during interval  $k$

- $\delta_{apk}^{rs}(t)$  time-dependent arc-path indicator variable; for each OD pair  $rs$ , set indicator  $\delta_{apk}^{rs}(t) = 1$  if time-dependent link  $(a,t)$  is in time-dependent route  $(p,k)$ ; otherwise,  $\delta_{apk}^{rs}(t) = 0$
- $\bar{\delta}_{apk}^{rs}(t)$  value of time-dependent arc-path indicator variable  $\delta_{apk}^{rs}(t)$  is estimated
- $\varepsilon$  convergence criterion
- $\gamma$  predetermined move size
- $\hat{\pi}^{rs}(k)$  minimal generalized route travel time between O-D pair  $rs$  during time interval  $k$
- $\tau_a(t)$  actual travel time for link  $a$  during time interval  $t$
- $L(\mathbf{u}, \hat{\pi}, \beta)$  Lagrangian expressed in terms of inflows and Lagrange [Lagrangian ?] multipliers
- $\Omega_{cf}$  side constrained feasible region, which includes FIFO and link capacity constraints
- \*
- $\chi_{2a}(t, t + \Delta t)$  time-dependent arc-arc indicator variable; set indicator  $\chi_{2a}(t, t + \Delta t) = 1$  if both time-dependent link flows  $u_a(t) > 0$  and  $u_a(t + \Delta t) > 0$ ; otherwise,  $\chi_{2a}(t, t + \Delta t) = 0$

## APPENDIX II: Equivalence Analysis

An equivalence analysis is conducted in the following.

**Theorem 1:** Under a certain propagation relationship  $\{\delta_{apk}^{rs}(t)\} = \{\bar{\delta}_{apk}^{rs}(t)\}$ , equilibrium conditions (5) imply variational inequality (13) and vice versa.

**Proof of necessity:** Under a certain propagation relationship  $\{\delta_{apk}^{rs}(t)\} = \{\bar{\delta}_{apk}^{rs}(t)\}$ , user equilibrium conditions (5) can be reformulated as VIP (13). By using Eqn (7), Eqn (5) can be rewritten as

$$\left[ c_p^{rs*}(k) + \beta_{1p}^{rs*}(k) + \beta_{2p}^{rs*}(k) - \bar{\pi}^{rs}(k) \right] h_p^{rs*}(k) = 0 \quad \forall r, s, p, k \quad (33)$$

By adopting Eqns (5) and (18), we have

$$\left[ c_p^{rs*}(k) + \beta_{1p}^{rs*}(k) + \beta_{2p}^{rs*}(k) - \bar{\pi}^{rs}(k) \right] h_p^{rs}(k) \geq 0 \quad \forall r, s, p, k \quad (34)$$

Subtracting Eqn (32) from Eqn (33) results in:

$$\left[ c_p^{rs*}(k) + \beta_{1p}^{rs*}(k) + \beta_{2p}^{rs*}(k) - \bar{\pi}^{rs}(k) \right] \left[ h_p^{rs}(k) - h_p^{rs*}(k) \right] \geq 0 \quad \forall r, s, p, k \quad (35)$$

Summing over  $r, s, p, k$  yields:

$$\begin{aligned} & \sum_{rs} \sum_p \sum_k \left[ c_p^{rs*}(k) + \beta_{1p}^{rs*}(k) + \beta_{2p}^{rs*}(k) \right] \left[ h_p^{rs}(k) - h_p^{rs*}(k) \right] \\ & - \sum_{rs} \sum_k \bar{\pi}^{rs}(k) \sum_p \left[ h_p^{rs}(k) - h_p^{rs*}(k) \right] \geq 0 \end{aligned} \quad (36)$$

By making the substitution of  $\bar{q}^{rs}(k)$  for both  $\sum_p h_p^{rs}(k)$  and  $\sum_p h_p^{rs*}(k)$ , the second term vanishes; the remaining term results in the following VIP:

$$\sum_{rs} \sum_p \sum_k \left[ c_p^{rs*}(k) + \beta_{1p}^{rs*}(k) + \beta_{2p}^{rs*}(k) \right] \left[ h_p^{rs}(k) - h_p^{rs*}(k) \right] \geq 0 \quad (37)$$

By applying Eqns (8)~(10), one obtains:

$$\sum_{rs} \sum_p \sum_k \left[ \sum_a \sum_t \left( c_a^*(t) + \beta_{1a}^*(t) + \beta_{2a}^*(t) \right) \delta_{apk}^{rs*}(t) \right] \left[ h_p^{rs}(k) - h_p^{rs*}(k) \right] \geq 0 \quad (38)$$

By changing the order of the summation, it follows that:

$$\sum_a \sum_t \left( c_a^*(t) + \beta_{1a}^*(t) + \beta_{2a}^*(t) \right) \sum_{rs} \sum_p \sum_k \delta_{apk}^{rs*}(t) \left[ h_p^{rs}(k) - h_p^{rs*}(k) \right] \geq 0 \quad (39)$$

By adopting the definition  $u_a(t) = \sum_{rs} \sum_p \sum_k h_p^{rs}(k) \delta_{apk}^{rs}(t)$ , we have:

$$\sum_a \sum_t c_a^*(t) \left[ u_a(t) - u_a^*(t) \right] + \beta_{1a}^*(t) \left[ u_a(t) - u_a^*(t) \right] + \beta_{2a}^*(t) \left[ u_a(t) - u_a^*(t) \right] \geq 0 \quad (40)$$

Through the link capacity side constraint and its corresponding complementarity slackness condition,  $\beta_{1a}^*(t)[u_a^*(t) - CAP_a(t)] = 0$ , one obtains

$$\begin{aligned} & \beta_{1a}^*(t)[u_a(t) - u_a^*(t)] \\ &= \beta_{1a}^*(t)[u_a(t) - CAP_a(t)] - \beta_{1a}^*(t)[u_a^*(t) - CAP_a(t)] \\ &= \beta_{1a}^*(t)[u_a(t) - CAP_a(t)] \leq 0 \end{aligned} \tag{41}$$

By introducing incidence variables into the FIFO side constraint and using its corresponding complementarity slackness condition,  $\beta_{2a}^*(t, t + \Delta t)[c_a^*(t) - \Delta t - c_a(t + \Delta t)] = 0, \forall a, t, \Delta t = 1, 2, 3 \dots, |T| - t, u_a(t) > 0, u_a(t + \Delta t) > 0$ , we have

$$\begin{aligned} & \beta_{2a}^*(t)[u_a(t) - u_a^*(t)] \\ &= \left[ \chi_{2a}(t, t + 1)\beta_{2a}(t, t + 1) + \chi_{2a}(t, t + 2)\beta_{2a}(t, t + 2) \right. \\ & \quad \left. + \dots + \chi_{2a}(t, |T|)\beta_{2a}(t, |T|) \right] c_a'(t)[u_a(t) - u_a^*(t)] \\ &= \left[ \chi_{2a}(t, t + 1)\beta_{2a}(t, t + 1) + \chi_{2a}(t, t + 2)\beta_{2a}(t, t + 2) \right. \\ & \quad \left. + \dots + \chi_{2a}(t, |T|)\beta_{2a}(t, |T|) \right] [c_a(t) - c_a^*(t)] \\ &= \left[ \chi_{2a}(t, t + 1)\beta_{2a}(t, t + 1) + \chi_{2a}(t, t + 2)\beta_{2a}(t, t + 2) \right. \\ & \quad \left. + \dots + \chi_{2a}(t, |T|)\beta_{2a}(t, |T|) \right] [c_a(t) - \Delta t - c_a(t + \Delta t)] \\ & \quad - \left[ \chi_{2a}(t, t + 1)\beta_{2a}(t, t + 1) + \chi_{2a}(t, t + 2)\beta_{2a}(t, t + 2) \right. \\ & \quad \left. + \dots + \chi_{2a}(t, |T|)\beta_{2a}(t, |T|) \right] [c_a^*(t) - \Delta t - c_a(t + \Delta t)] \\ &= \left[ \chi_{2a}(t, t + 1)\beta_{2a}(t, t + 1) + \chi_{2a}(t, t + 2)\beta_{2a}(t, t + 2) \right. \\ & \quad \left. + \dots + \chi_{2a}(t, |T|)\beta_{2a}(t, |T|) \right] [c_a(t) - \Delta t - c_a(t + \Delta t)] \leq 0 \end{aligned} \tag{42}$$

By using Eqns (40) and (41), Eqn (39) can be reduced to

$$\sum_a \sum_t c_a^*(t)[u_a(t) - u_a^*(t)] \geq 0 \quad \forall \mathbf{u} \in \Omega_{cf}^* \tag{43}$$

The above inequality is identical to VIP (13).

**Proof of sufficiency:** We next prove that VIP (13) can induce dynamic equilibrium conditions (5). If vector  $\mathbf{u}^* \in \Omega_{cf}^* = \{\mathbf{u} \in R^n | \mathbf{f}(\mathbf{u}) \geq \mathbf{0}, \mathbf{e}(\mathbf{u}) = \mathbf{0}\}$  is a solution to VI (13) and gradient  $\nabla \mathbf{f}_i(\mathbf{u}^*), \forall i$  such that  $\mathbf{f}_i(\mathbf{u}^*) = \mathbf{0}, \forall i$  and  $\nabla \mathbf{e}_i(\mathbf{u}^*), \forall i$  are linear independent, then, following to Tobin (1986),  $\mathbf{u}^*$  also solves the following nonlinear programming problem (linear objective function).

$$\min_{\mathbf{u} \in \bar{\Omega}_{cf}} z(\mathbf{u}) = \sum_a \sum_t c_a(t) (u_a(t) - u_a^*(t)) \quad (44)$$

where the feasible region  $\bar{\Omega}_{cf}$  is a subset of  $\Omega_{cf}$ , and is delineated by the following constraints:

Flow conservation constraint:

$$\sum_p h_p^{rs}(k) = \bar{q}^{rs}(k) \quad \forall r, s, k \quad (45)$$

Nonnegativity constraint:

$$h_p^{rs}(k) \geq 0 \quad \forall r, s, p, k \quad (46)$$

Definitional constraints:

$$u_a(t) = \sum_{rs} \sum_p \sum_k h_p^{rs}(k) \bar{\delta}_{apk}^{rs}(t) \quad \forall a, t \quad (47)$$

$$c_p^{rs}(k) = \sum_a \sum_t c_a(t) \bar{\delta}_{apk}^{rs}(t) \quad \forall r, s, p, k \quad (48)$$

$$\bar{\delta}_{apk}^{rs}(t) = \{0, 1\} \quad \forall r, s, a, p, k, t \quad (49)$$

Link capacity constraint:

$$u_a(t) \leq CAP_a(t) \quad \forall a, t \quad (50)$$

First-in-first-out constraint:

$$c_a(t) \leq \Delta t + c_a(t + \Delta t) \quad \forall a, t, \Delta t = 1, 2, 3, \dots, |T| - t, u_a(t) > 0, u_a(t + \Delta t) > 0 \quad (51)$$

The Lagrangian can be constructed by relaxing flow conservation, link capacity and FIFO constraints. Introducing the corresponding dual variables  $\{\bar{\pi}^{rs}(k)\}$ ,  $\{\beta_{1a}(t)\}$  and  $\{\beta_{2a}(t, t + \Delta t)\}$ , respectively, we have:

$$\begin{aligned}
 L(\mathbf{h}, \hat{\pi}, \beta) &= \sum_a \sum_t \int_0^{u_a(t)} c_a(\mathbf{u} \setminus u_a(t), \omega) d\omega + \sum_{rs} \sum_k \hat{\pi}^{rs}(k) \left[ \bar{q}^{rs}(k) - \sum_p h_p^{rs}(k) \right] \\
 &+ \sum_a \sum_t \beta_{1a}(t) [u_a(t) - CAP_a(t)] \\
 &+ \sum_a \sum_t \sum_{\Delta t=1}^{|T|-t} \chi_{2a}(t, t + \Delta t) \beta_{2a}(t, t + \Delta t) [c_a(t) - \Delta t - c_a(t + \Delta t)]
 \end{aligned} \tag{52}$$

where  $\{\chi_{2a}(t, t + \Delta t)\}$  represents incident variables, first introduced in the literature and defined as follows:

$$\chi_{2a}(t, t + \Delta t) \begin{cases} = 1, & \text{if } u_a(t) > 0 \text{ and } u_a(t + \Delta t) > 0 \\ = 0, & \text{otherwise} \end{cases} \tag{53}$$

Dual variables  $\{\beta_{1a}(t)\}$  and  $\{\beta_{2a}(t, t + \Delta t)\}$  may have multiple values (Larsson et al., 1999). However, they will not affect the validation of our results if they are only regarded as virtual costs or tolls. The above Lagrangian function can be expanded as follows:

$$\begin{aligned}
 L(\mathbf{h}, \hat{\pi}, \beta) &= \sum_a \sum_t \int_0^{u_a(t)} c_a(\mathbf{u} \setminus u_a(t), \omega) d\omega + \sum_{rs} \sum_k \hat{\pi}^{rs}(k) \left[ \bar{q}^{rs}(k) - \sum_p h_p^{rs}(k) \right] \\
 &+ \sum_a \sum_t \beta_{1a}(t) [u_a(t) - CAP_a(t)] \\
 &+ \sum_a \sum_t \left\{ \begin{aligned} &\chi_{2a}(t, t + 1) \beta_{2a}(t, t + 1) [c_a(t) - 1 - c_a(t + 1)] \\ &+ \chi_{2a}(t, t + 2) \beta_{2a}(t, t + 2) [c_a(t) - 2 - c_a(t + 2)] \\ &+ \dots + \chi_{2a}(t, |T|) \beta_{2a}(t, |T|) [c_a(t) - (|T| - t) - c_a(|T|)] \end{aligned} \right\}
 \end{aligned} \tag{54}$$

The optimality conditions can thus be obtained by taking partial derivatives of the Lagrangian with respect to both primal and dual decision variables. Taking partial derivatives of the Lagrangian with respect to path flows yields:

$$\begin{aligned}
 \frac{\partial L(\mathbf{h}, \hat{\pi}, \beta)}{\partial h_p^{rs}(k)} &= c_p^{rs}(k) - \hat{\pi}^{rs}(k) + \sum_a \sum_t \beta_{1a}(t) \bar{\delta}_{apk}^{rs}(t) \\
 &+ \sum_a \sum_t \left[ \chi_{2a}(t, t + 1) \beta_{2a}(t, t + 1) + \chi_{2a}(t, t + 2) \beta_{2a}(t, t + 2) \right. \\
 &\quad \left. + \dots + \chi_{2a}(t, |T|) \beta_{2a}(t, |T|) \right] c_a'(t) \bar{\delta}_{apk}^{rs}(t) \geq 0 \quad \forall r, s, p, k
 \end{aligned} \tag{55}$$

where  $c'_a(t)$  is defined as the partial derivatives of  $c_a(t)$  with respect to  $u_a(t)$

$$c'_a(t) = \frac{\partial c_a(t)}{\partial u_a(t)} \quad \forall a, t \tag{56}$$

By complementarity slackness, we have

$$h_p^{rs}(k) \left( \begin{array}{l} c_p^{rs}(k) - \bar{\pi}^{rs}(k) + \sum_a \sum_t \beta_{1a}(t) \bar{\delta}_{apk}^{rs}(t) \\ + \sum_a \sum_t \left[ \chi_{2a}(t, t+1) \beta_{2a}(t, t+1) + \chi_{2a}(t, t+2) \beta_{2a}(t, t+2) \right] c'_a(t) \bar{\delta}_{apk}^{rs}(t) \\ + \dots + \chi_{2a}(t, |T|) \beta_{2a}(t, |T|) \end{array} \right) = 0 \quad \forall r, s, p, k \tag{57}$$

By using definitional Eqn (11), Eqns (54) and (56) become

$$c_p^{rs}(k) - \bar{\pi}^{rs}(k) + \sum_a \sum_t \beta_{1a}(t) \bar{\delta}_{apk}^{rs}(t) + \sum_a \sum_t \beta_{2a}(t) c'_a(t) \bar{\delta}_{apk}^{rs}(t) \geq 0 \quad \forall r, s, p, k \tag{58}$$

$$h_p^{rs}(k) \left[ c_p^{rs}(k) - \bar{\pi}^{rs}(k) + \sum_a \sum_t \beta_{1a}(t) \bar{\delta}_{apk}^{rs}(t) + \sum_a \sum_t \beta_{2a}(t) c'_a(t) \bar{\delta}_{apk}^{rs}(t) \right] = 0 \quad \forall r, s, p, k \tag{59}$$

By using Eqns (7)~(10), the above two equations can be simplified as

$$\bar{c}_p^{rs}(k) - \bar{\pi}^{rs}(k) \geq 0 \quad \forall r, s, p, k \tag{60}$$

$$h_p^{rs}(k) [\bar{c}_p^{rs}(k) - \bar{\pi}^{rs}(k)] = 0 \quad \forall r, s, p, k \tag{61}$$

In addition, taking partial derivatives of the Lagrangian with respect to dual variables results in Eqns (14) and (21)~(22). Assuming the independence of the gradients of the binding constraints is a sufficient condition for the Karush-Kuhn-Tucker (KKT) constraint qualification to be satisfied at  $\mathbf{u}^*$ . Therefore, by the KKT necessity theorem, there exist  $\beta$  and  $\pi$  such that the equilibrium conditions corresponding to the DUE-CF problem result. This completes the proof.

## APPENDIX III: Gradient Projection Method

The GP algorithm iterates between the original master problem (MP) and the restricted master problem (RMP). In the MP, for each O-D pair and time interval, a new shortest path is searched over the time-space network and added, if appropriate, to the shortest path set. The RMP is then invoked. In the RMP, the path flows associated with all shortest paths stored in the shortest path set are optimally determined. This procedure continues until the convergence criterion is met. The steps of the GP algorithm can be described as follows.

### The GP Algorithm

#### Step 0: Initialization.

Let  $n=0$ . For each O-D pair and time interval, search for a shortest route based on the free flow travel times  $\{c_{a_0}(t)\}$ , Lagrange multipliers  $\{\bar{\beta}_{1a}(t)\}$  and  $\{\bar{\beta}_{2a}(t, t + \Delta t)\}$ . Create a path set to store all shortest routes with path flows denoted as  $\{h_p^{rs}(k)^n\}$ .

#### Step 1: Master Problem.

Step 1.1: Let  $n=n+1$ . Update the link travel times  $\{c_a^n(t)\}$  based on path flows  $\{h_p^{rs}(k)^n\}$ .

Step 1.2: For each O-D pair and time interval, search for a new shortest path over the network based on  $\{c_a^n(t)\}$ ,  $\{\bar{\beta}_{1a}(t)\}$  and  $\{\bar{\beta}_{2a}(t, t + \Delta t)\}$ . The path set is augmented by the routes not contained in the set already. For each O-D pair and time interval, label the newest found or rediscovered time-dependent shortest route as  $(\hat{p}, \hat{k})$ .

#### Step 2: Restricted Master Problem.

Use the GP algorithm to solve the restricted master problem of Eqn (23).

Step 2.1: For each O-D pair and time interval, update the path flows  $\{h_p^{rs}(k)^{n+1}\}$  and the associated inflow pattern  $\{u_a^{n+1}(t)\}$  by the following formulas:

$$h_p^{rs}(k)^{n+1} = \max\{0, h_p^{rs}(k)^n - \alpha_p^{rs}(k)^n d_p^{rs}(k)^n\} \quad \forall r, s, (p, k) \neq (\hat{p}, \hat{k}) \quad (62)$$

$$h_{\hat{p}}^{rs}(\hat{k})^{n+1} = \bar{q}^{rs}(k) - \sum_{\substack{(p,k) \\ \neq (\hat{p},\hat{k})}} h_p^{rs}(k)^{n+1} \quad \forall r, s \quad (63)$$

$$u_a^{n+1}(t) = \sum_{rs} \sum_p \sum_k h_p^{rs}(k)^{n+1} \bar{\delta}_{apk}^{rs}(t) \quad \forall a, t \quad (64)$$

where

$$d_p^{rs}(k) = \bar{c}_{\hat{p}}^{rs}(k) - \bar{c}_p^{rs}(k) \quad \forall r, s, (p, k) \neq (\hat{p}, \hat{k}) \quad (65)$$

$$\alpha_p^{rs}(k) = \frac{1}{\sum_{(a,t) \in A} \bar{c}'_a(t) + \sum_{(a,t) \in \hat{A}} \bar{c}'_a(t)} \quad \forall r, s, (p, k) \neq (\hat{p}, \hat{k}) \quad (66)$$

$$A = \left\{ (a, t) \mid (a, t) \in \left\{ \left( \bar{\delta}_{apk}^{rs}(t) = 1 \right) \wedge \left( \bar{\delta}_{\hat{p}\hat{k}}^{rs}(t) = 0 \right) \right\} \right\} \quad (67)$$

$$\hat{A} = \left\{ (a, t) \mid (a, t) \in \left\{ \left( \bar{\delta}_{apk}^{rs}(t) = 0 \right) \wedge \left( \bar{\delta}_{\hat{p}\hat{k}}^{rs}(t) = 1 \right) \right\} \right\} \quad (68)$$

$$\alpha_{\max} \geq \alpha_p^{rs}(k) \geq 0 \quad \forall r, s, (p, k) \neq (\hat{p}, \hat{k}) \quad (69)$$

Step 2.2: Calculate the difference between the link inflow patterns in two successive iterations by the following formula:

$$\varepsilon = \max_{a,t} \left| \frac{u_a^{n+1}(t) - u_a^n(t)}{u_a^n(t)} \right| \leq 0.0001 \quad (70)$$

If the difference  $\varepsilon$  is less than the predetermined tolerance, the updated solution is deemed optimal. Otherwise, go to Step 1.

In Eqn (64), the search direction is determined by the negative first derivative of the objective function (23) with respect to non-shortest route flow. In Eqn (65) the move size  $\{\alpha_p^{rs}(k)\}$  is determined by the inverse of the second derivative of the objective function (23) with respect to non-shortest route flow. (The reader may refer to Jayakrishnan et al. (1994) for detailed derivations.)

In Step 2.1, the opposite search direction,  $(\hat{c}_p^{rs}(k) - \hat{c}_p^{rs}(\hat{k}))$ , in Eqn (64) is determined by the first derivative of the objective function (23) with respect to non-shortest route flow, as follows:

$$\begin{aligned}
 \frac{\partial \mathcal{L}(\mathbf{h}, \boldsymbol{\beta})}{\partial h_p^{rs}(k)} &= \frac{\partial \sum_a \sum_t \int_0^{u_a(t)} c_a(\mathbf{u} \setminus u_a(t), \omega) d\omega}{\partial h_p^{rs}(k)} + \frac{\partial \sum_a \sum_t \beta_{1a}(t) [u_a(t) - CAP_a(t)]}{\partial h_p^{rs}(k)} \\
 &\quad + \frac{\partial \sum_a \sum_t \sum_{\Delta t=1}^{|T|-t} \chi_{2a}(t, t + \Delta t) \beta_{2a}(t, t + \Delta t) [c_a(t) - \Delta t - c_a(t + \Delta t)]}{\partial h_p^{rs}(k)} \\
 &= \sum_a \sum_t \frac{\partial \int_0^{u_a(t)} c_a(\mathbf{u} \setminus u_a(t), \omega) d\omega}{\partial u_a(t)} \frac{\partial u_a(t)}{\partial h_p^{rs}(k)} + \frac{\partial \sum_a \sum_t \beta_{1a}(t) [u_a(t) - CAP_a(t)]}{\partial u_a(t)} \frac{\partial u_a(t)}{\partial h_p^{rs}(k)} \\
 &\quad + \frac{\partial \sum_a \sum_t \sum_{\Delta t=1}^{|T|-t} \chi_{2a}(t, t + \Delta t) \beta_{2a}(t, t + \Delta t) [c_a(t) - \Delta t - c_a(t + \Delta t)]}{\partial u_a(t)} \frac{\partial u_a(t)}{\partial h_p^{rs}(k)} \\
 &= \sum_a \sum_t \left( c_a(t) + \beta_{1a}(t) + \sum_{\Delta t=1}^{|T|-t} \chi_{2a}(t, t + \Delta t) \beta_{2a}(t, t + \Delta t) c'_a(t) \right) \frac{\partial \sum_{r's' \in (p', k')} h_{p', s'}^{r's'}(k) \bar{\delta}_{ap'k'}^{r's'}(t)}{\partial h_p^{rs}(k)} \\
 &= \sum_a \sum_t \hat{c}_a(t) \frac{\partial \sum_{r's' \in (p', k')} h_{p', s'}^{r's'}(k) \bar{\delta}_{ap'k'}^{r's'}(t)}{\partial h_p^{rs}(k)} \quad \forall r, s, (p, k) \neq (\hat{p}, \hat{k}) \tag{71}
 \end{aligned}$$

where

$$\hat{c}_a(t) = c_a(t) + \beta_{1a}(t) + \beta_{2a}(t) \quad \forall a, t \tag{72}$$

and

$$\beta_{2a}(t) = \sum_{\Delta t=1}^{|T|-t} \chi_{2a}(t, t + \Delta t) \beta_{2a}(t, t + \Delta t) c'_a(t) \quad \forall a, t \tag{73}$$

For each O-D pair, the latest added time-dependent shortest path is labeled  $(\hat{p}, \hat{k})$  in the path set. The corresponding flow associated with this shortest path  $(\hat{p}, \hat{k})$  can be expressed in terms of the other paths in the path set, according to the flow conservation constraint,

that is,  $h_{\hat{p}}^{rs}(\hat{k}) = \bar{q}^{rs} - \sum_{(p,k) \neq (\hat{p}, \hat{k})} h_p^{rs}(k)$ ,  $\forall r, s$ . Subsequently, Eqn (70) can be further rearranged as follows:

$$\begin{aligned}
\frac{\partial L(\mathbf{h}, \beta)}{\partial h_p^{rs}(k)} &= \sum_a \sum_t \hat{c}_a(t) \frac{\partial \sum_{r's'} \sum_{(p',k')} h_{p'}^{r's'}(k) \bar{\delta}_{ap'k'}^{r's'}(t)}{\partial h_p^{rs}(k)} \\
&= \sum_a \sum_t \hat{c}_a(t) \frac{\partial \sum_{r's'} \left( h_{\hat{p}}^{r's'}(\hat{k}) \bar{\delta}_{a\hat{p}\hat{k}}^{r's'}(t) + \sum_{\substack{(p',k') \\ \neq (\hat{p}, \hat{k})}} h_{p'}^{r's'}(k) \bar{\delta}_{ap'k'}^{r's'}(t) \right)}{\partial h_p^{rs}(k)} \\
&= \sum_a \sum_t \hat{c}_a(t) \frac{\partial \sum_{r's'} \left( \left( \bar{q}^{r's'}(k') - \sum_{\substack{(p',k') \\ \neq (\hat{p}, \hat{k})}} h_{p'}^{r's'}(k') \right) \bar{\delta}_{a\hat{p}\hat{k}}^{r's'}(t) + \sum_{\substack{(p',k') \\ \neq (\hat{p}, \hat{k})}} h_{p'}^{r's'}(k) \bar{\delta}_{ap'k'}^{r's'}(t) \right)}{\partial h_p^{rs}(k)} \tag{74} \\
&= \sum_a \sum_t \hat{c}_a(t) (\bar{\delta}_{apk}^{rs}(t) - \bar{\delta}_{a\hat{p}\hat{k}}^{rs}(t)) \\
&= \hat{c}_p^{rs}(k) - \hat{c}_{\hat{p}}^{rs}(\hat{k}) \quad \forall r, s, (p, k) \neq (\hat{p}, \hat{k})
\end{aligned}$$

The move size  $\{\alpha_p^{rs}(k)\}$  used in Eqn (65) is determined by the inverse of the second derivative of the objective function (23) with respect to non-shortest route flow, i.e.,

$\alpha_p^{rs}(k) = \left( \frac{\partial^2 L(\mathbf{h}, \beta)}{\partial h_p^{rs}(k)^2} \right)^{-1}$ ,  $\forall r, s, (p, k) \neq (\hat{p}, \hat{k})$ . The second derivative is computed as follows:

lows:

$$\begin{aligned}
 \frac{\partial^2 L(\mathbf{h}, \beta)}{\partial h_p^{rs}(k)^2} &= \frac{\partial(\hat{c}_p^{rs}(k) - \hat{c}_{\hat{p}}^{rs}(k))}{\partial h_p^{rs}(k)} = \frac{\partial\left(\sum_a \sum_t \hat{c}_a(t) \bar{\delta}_{apk}^{rs}(t) - \sum_a \sum_t \hat{c}_a(t) \bar{\delta}_{a\hat{p}k}^{rs}(t)\right)}{\partial h_p^{rs}(k)} \\
 &= \frac{\partial \sum_a \sum_t \hat{c}_a(t) \bar{\delta}_{apk}^{rs}(t)}{\partial u_a(t)} \frac{\partial u_a(t)}{\partial h_p^{rs}(k)} - \frac{\partial \sum_a \sum_t \hat{c}_a(t) \bar{\delta}_{a\hat{p}k}^{rs}(t)}{\partial u_a(t)} \frac{\partial u_a(t)}{\partial h_p^{rs}(k)} \\
 &= \sum_a \sum_t \frac{\partial \hat{c}_a(t) \bar{\delta}_{apk}^{rs}(t)}{\partial u_a(t)} \frac{\partial u_a(t)}{\partial h_p^{rs}(k)} - \partial \sum_a \sum_t \frac{\partial \hat{c}_a(t) \bar{\delta}_{a\hat{p}k}^{rs}(t)}{\partial u_a(t)} \frac{\partial u_a(t)}{\partial h_p^{rs}(k)} \\
 &= \sum_a \sum_t \hat{c}'_a(t) \bar{\delta}_{apk}^{rs}(t) \frac{\partial \sum_{r's'} \sum_{(p',k')} h_{p'}^{r's'}(k') \bar{\delta}_{ap'k'}^{r's'}(t)}{\partial h_p^{rs}(k)} \\
 &\quad - \sum_a \sum_t \hat{c}'_a(t) \bar{\delta}_{a\hat{p}k}^{rs}(t) \frac{\partial \sum_{r's'} \sum_{(p',k')} h_{p'}^{r's'}(k') \bar{\delta}_{ap'k'}^{r's'}(t)}{\partial h_p^{rs}(k)} \\
 &= \sum_a \sum_t \hat{c}'_a(t) \bar{\delta}_{apk}^{rs}(t) \frac{\partial(h_p^{rs}(k) \bar{\delta}_{apk}^{rs}(t) + h_{\hat{p}}^{rs}(\hat{k}) \bar{\delta}_{a\hat{p}k}^{rs}(t))}{\partial h_p^{rs}(k)} \\
 &\quad - \sum_a \sum_t \hat{c}'_a(t) \bar{\delta}_{a\hat{p}k}^{rs}(t) \frac{\partial(h_p^{rs}(k) \bar{\delta}_{apk}^{rs}(t) + h_{\hat{p}}^{rs}(\hat{k}) \bar{\delta}_{a\hat{p}k}^{rs}(t))}{\partial h_p^{rs}(k)} \\
 &= \sum_a \sum_t \hat{c}'_a(t) \bar{\delta}_{apk}^{rs}(t) \frac{\partial\left(h_p^{rs}(k) \bar{\delta}_{apk}^{rs}(t) + \left(\bar{q}^{rs}(k) - \sum_{(p',k') \neq (\hat{p}, \hat{k})} h_{p'}^{rs}(k')\right) \bar{\delta}_{a\hat{p}k}^{rs}(t)\right)}{\partial h_p^{rs}(k)} \\
 &\quad - \sum_a \sum_t \hat{c}'_a(t) \bar{\delta}_{a\hat{p}k}^{rs}(t) \frac{\partial\left(h_p^{rs}(k) \bar{\delta}_{apk}^{rs}(t) + \left(\bar{q}^{rs}(k) - \sum_{(p',k') \neq (\hat{p}, \hat{k})} h_{p'}^{rs}(k')\right) \bar{\delta}_{a\hat{p}k}^{rs}(t)\right)}{\partial h_p^{rs}(k)} \\
 &= \sum_a \sum_t \hat{c}'_a(t) \bar{\delta}_{apk}^{rs}(t) (\bar{\delta}_{apk}^{rs}(t) - \bar{\delta}_{a\hat{p}k}^{rs}(t)) \\
 &\quad - \sum_a \sum_t \hat{c}'_a(t) \bar{\delta}_{a\hat{p}k}^{rs}(t) (\bar{\delta}_{apk}^{rs}(t) - \bar{\delta}_{a\hat{p}k}^{rs}(t)) \\
 &= \sum_a \sum_t (\hat{c}'_a(t) \bar{\delta}_{apk}^{rs}(t) + \hat{c}'_a(t) \bar{\delta}_{a\hat{p}k}^{rs}(t)) - \sum_a \sum_t 2\hat{c}'_a(t) \bar{\delta}_{apk}^{rs}(t) \bar{\delta}_{a\hat{p}k}^{rs}(t) \\
 &= \sum_{(a,t) \in A} \hat{c}'_a(t) + \sum_{(a,t) \in \hat{A}} \hat{c}'_a(t)
 \end{aligned} \tag{75}$$

where  $\hat{c}'_a(t) = \frac{\partial c_a(t)}{\partial u_a(t)}$ . The step size thus becomes:

$$\alpha_p^{rs}(k) = \frac{1}{\sum_{(a,t) \in A} \hat{c}'_a(t) + \sum_{(a,t) \in \hat{A}} \hat{c}'_a(t)} \quad \forall r, s, (p, k) \neq (\hat{p}, \hat{k}) \quad (76)$$

$$A = \left\{ (a, t) \mid (a, t) \in \left\{ \left( \bar{\delta}_{apk}^{rs}(t) = 1 \right) \wedge \left( \bar{\delta}_{ap\hat{k}}^{rs}(t) = 0 \right) \right\} \right\} \quad (77)$$

$$\hat{A} = \left\{ (a, t) \mid (a, t) \in \left\{ \left( \bar{\delta}_{apk}^{rs}(t) = 0 \right) \wedge \left( \bar{\delta}_{ap\hat{k}}^{rs}(t) = 1 \right) \right\} \right\} \quad (78)$$

$$\alpha_{\max} \geq \alpha_p^{rs}(k) \geq 0 \quad \forall r, s, (p, k) \neq (\hat{p}, \hat{k})$$