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Perfect Equilibria of Network Flows

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Abstract: In the paper, we enhance the notion of user equilibrium of network flows in a way to take account of group behaviour of network users. We present some initial existence results for the enhanced equilibrium notion and their implications for the equilibrium economics and dynamics of transport markets.

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1 Introduction

Transport market equilibrium models have been intensively researched since the early fifties of the past century, when their computer versions first appeared in American transportation studies, e.g. [6], [1], [2]. These models describe various equilibrium mechanisms, by which transport markets determine prices of their transport services.

Restricted versions of these models assume that the supply side of the transport market is given and fixed, which amounts to working with a model, including a transport network, describing transport services of the market, and their users, trying to minimize their individual transport costs. The user equilibrium of the network is then defined as that state or distribution of network flows, in which no individual user can improve his bargaining position on the market by decreasing his individual transport costs.

In the past, a lot of these models resorted to the "continuous" paradigm of Walrasian equilibrium models: they assumed that there is a continuous functional relationship between network flow volumes and unit transport costs. In such a setting, it is relatively safe to define the user equilibrium notion by means of incremental changes of values of unit transport cost functions, caused by actions and decisions of individual economic agents, e.g. [5].

In recent years, many researchers, analyzing road and pedestrian traffic, abandoned the traditional network flow setting in favour of self driven many particle systems, e.g. [3]. The dynamic description of these systems incorporates intensive interactions between individual units of flow on links of road and pedestrian transport networks and leads to many useful insights into the dynamics of traffic flows. As we are interested also in the dynamics of transport markets, for which such interactions on the micro level of the market are not so pronounced or are simply less important in economic terms, we will, throughout this paper, use the traditional network flow equilibrium setting.

It is clear that the above notion of user equilibrium does not take account of eventual cooperative behaviour of economic agents and must be adapted, if one wants to analyze its dynamic consequences. In this paper, we will be interested only in "selfish" type of cooperative behaviour of economic agents: this type of behaviour implies consciously coordinated actions and/or decisions of a group of economic agents, which are to the benefit, or to the profit, of each of the individual members of the group. In other words, we will exclude the cooperative behaviour of groups of economic agents, requiring system optimization of the actions of the members of such a group.

In dealing with the dynamic effects of consciously coordinated cooperative actions of economic agents one has to take account also of similar dynamic effects of non-coordinated group behaviour of economic agents.

If, in the traditional network flow setting, one acknowledges the fact that, with rare exceptions, transport flows consist of indivisible discrete units, such as individual passengers, parcels, vehicle loads, etc., and drops the continuous paradigm, one ends up searching for integer solutions to equilibrium problems, in which unit transport costs are often not continuous functions of flow volumes. In dealing with such problems, the traditional incremental approach to the definition of equilibrium turns out to be too crude a tool precisely because of possible group behaviour of economic agents. This can be probably best seen by analyzing a very simple example of this type.

1.1 Example

Let us assume that our transport network consists of a single origin-destination pair, connected by two alternative transport services, e.g., bus and taxi services, and three economic agents, using these services. Let the passenger price of the bus service p_1 be independent of the number of its passengers and equal 1. Let the passenger price of the taxi service p_2, on the other hand, depend on the number of its passengers, resulting in possible values $p_2(1) = 3/2$, $p_2(2) = 3/4$, and $p_2(3) = 1/3$.

Under the assumption that the economic agents are familiar with price functions and their values, this model has two conventional non-cooperative user equilibria.

In the first user equilibrium, all three economic agents use the bus service. In the second user equilibrium, all three economic agents use the taxi service.

The difference between these two equilibria is obvious at a first glance: the conventional equilibrium nature of the first user equilibrium is guaranteed only under the assumption that only one economic agent per a time can change to a new service. If one allows instead simultaneous moves of two or more economic agents, these moves turn out to be profitable for individual agents, and this flow distribution ceases to be in equilibrium. In contrast to the first equilibrium, this cannot happen in the second user equilibrium: regardless of the number of economic agents, considering moving to the first transport service, they would find it to be an unprofitable move for individual agents.

1.2 Group Behaviour of Economic Agents

As shown in the above example, the equilibrium nature of non-cooperative market equilibria can crucially depend on the possibility of simultaneous actions of groups of economic agents.

During the equilibrium process, the group behaviour of economic agents can arise from various sources. It can be the result of the above conscious cooperative behaviour of agents.

It can also be the result of system optimization of actions of groups of agents, e.g. in freight transport, which we will, as observed above, exclude from our observations.

But it can also be a matter of pure chance, caused by purely accidental coinciding, in space and time, of actions of economic agents. It is important to note that, in the case, when such coinciding of actions proves to be profitable for each of the economic agents, involved in such an event, it can become strong enough economic motivation for economic agents to repeat their actions. This would, of course, have the same dynamic and economic consequences as consciously coordinated cooperative actions of agents. To discern it from the conscious cooperative group behaviour, we will call this type of group behaviour *event driven group behaviour* of economic agents.

There is another possible source of group behaviour, sometimes described as "moving with the crowd": when an economic agent is faced with a market opportunity, which can be exploited only by simultaneous actions of group of economic agents, he can make an educated guess that other agents will try to exploit this opportunity. If this seems to be the case, he can simply "join the crowd" and exploit this opportunity together with other economic agents. This behaviour is, by itself, similar to the event driven group behaviour of agents, the only difference lies in conscious planning and/or predictive considerations of economic agents that lead to it (conscious planning and predictive considerations of agents can be, of course, also the result of their previous experiences with event driven group behaviour). We will call this type of group behaviour *predictive group behaviour of economic agents.*

The predictive type of group behaviour is, for example, rather common in stock markets: a bullish stock market investor will buy a security, if he expects that many other market participants are considering the same move, as that would hopefully increase the price of this security; a similar observation is valid for a bearish investor, considering short selling of a stock, and short selling of stocks by other participants.

Various type of group behaviour are common also in transport markets. In passenger transport, they are behind various innovative services, which have been, particularly in recent years, becoming increasingly popular in a number of European countries, e.g. sharing of a second car, sharing of taxi rides, flights at special discount rates, and taking place only, if there are enough of passengers, etc.. In freight transport, the cooperative behaviour represents one of the well known alternative strategies of smaller carriers in their competing with bigger carriers.

The group behaviour is a rather common phenomenon also in transport networks themselves: network users do tend to think in terms of network flows and not in terms of individual passengers or agents. This kind of thinking would often automatically lead to various types of group behaviour, ultimately resulting in moves of larger chunks of flows. Concerning this, it is interesting to observe that the group behaviour of network users can cause considerable problems in traffic optimization schemes, aimed at providing users of transport networks with real time data on traffic volumes and/or travelling conditions. Though such schemes are usually directed at individual users, the information provided becomes available to large number of users, which can, by itself, trigger also eventually undesirable types of reactions of larger groups of users.

It is the aim of this paper to adapt the traditional definition of user equilibrium of network flows in a way to allow for these types of group behaviour and to analyze basic existence, dynamic, and economic properties of the resulting equilibrium notion.

1.3 Structure of the Paper

In Section 2, we present the formal definition of the adapted equilibrium notion under the assumption of complete information on the part of economic agents.

In Section 3, we analyze the existence of equilibria in a number of two-dimensional additive linear models and discuss the implications of these examples for understanding the general equilibrium dynamics of transport markets.

In Section 4, we present two simple existence theorems for additive linear models, illustrating the well known tendency of transport markets towards all-or-nothing equilibrium solutions, when new and superior transport technologies arrive at these markets.

Section 5 discusses economic, dynamic and algorithmic implications of the new equilibrium notion. Section 6 presents concluding remarks and suggestions for further research of this equilibrium notion.

2 Definition of Equilibrium

2.1 Transport Network

To ease the formal definition of the equilibrium notion, we will assume a very simple structure of the transport market: it will consist of a single origin-destination node pair, and a finite set $\mathbb{P} = \{p_1, ..., p_m\}$ of alternative transport services or paths, connecting this pair.

There will be a finite set $\mathbb{E} = \{e\}$ of n economic agents, which will be, along a discrete or continuous time horizon $\mathbb{T}$, making regular trips between this origin and destination pair. In doing this, they will be trying to pick up services from the set $\mathbb{P}$ according to the rule, given below. The resulting combined decisions of agents will be described by mappings of the form $D : \mathbb{E} \to \mathbb{P}$, called *decision mappings*. In this context, $p = D(e)$ means that the economic agent e decides to use the alternative service or path p. The finite set $\mathbb{D} = \{D : \mathbb{E} \to \mathbb{P}\}$ of all possible decision mappings will be called *the decision space* of the equilibrium process.

We will assume that economic agents are trying to minimize their individual transport costs. These costs would include both the eventual price of a transport service and the monetary value of the time spent in consuming it. As, during the equilibrium process, both cost items can depend on the combined decisions of all economic agents, unit transport costs will be described by a function c of the form $p : \mathbb{D} \to \mathbb{R}^m$. The vector $(c_1(D), c_2(D), \ldots, c_m(D))^T \in \mathbb{R}^m$ of the m-dimensional Euclidean space will thus denote the unit costs connected with using respective transport services at the point $D \in \mathbb{D}$ of the decision space. We will not require that these unit costs take on only positive values - eventual negative values of these costs can be simply taken to imply that economic agents can make some profit in making their trips.

A remark concerning the functional forms of transport costs that can appear in practice is probably in place. As these costs include also the monetary value of time, spent in consuming a transport service, their dependence on the decisions of economic agents can be, in practice, quite complex and general. For example, in a congested urban transport network, bus services along a particular route affect transport times of private car users using the same route and vice versa. This would ultimately imply that respective transport costs of bus and private car users depend on the decisions of agents using both bus and private car services. In a similar manner, and in a congested railway network, used by both passenger and freight services, passenger services can affect transport times of freight services and quicker services can affect transport times of slower services (slower trains have to wait at stations or specially designated points of the network for quicker trains to pass). This would again lead to quite general types of the functional form of user transport costs.

The above functional form of transport costs is, in fact, and to a certain degree, a simplification of the actual conditions, prevailing in transport markets: as monetary value of time, spent in consuming transport services, varies from economic agent to economic agent, one would theoretically have to assume one such function for each individual economic agent. In this paper, we shall, however, avoid this technical complication.

2.2 Formal Definition

If E_1 and E_2 are two disjoint subsets of the set E of economic agents, and D_1 and D_2 mappings of the form $D_1 : E_1 \to \mathbb{P}$ and $D_2 : E_2 \to \mathbb{P}$ respectively, let us denote by $D_1 \biguplus D_2$ the mapping of the form $D_1 \biguplus D_2 : E_1 \bigcup E_2 \to \mathbb{P}$, defined by $(D_1 \biguplus D_2)(e) = D_1(e)$ for $e \in E_1$, and by $(D_1 \biguplus D_2)(e) = D_2(e)$ for $e \in E_2$. Note that, in the case of complementary sets E_1 and E_2 with $E_1 \bigcup E_2 = \mathbb{E}$, the mapping $D_1 \biguplus D_2$ is an element of the decision space $\mathbb{D}$.

Definition. *Let D be a point of the decision space $\mathbb{D}$. We will say that a subset $G \subset \mathbb{E}$ of economic agents can improve their bargaining position at this point, if there exists a mapping $H : G \to \mathbb{P}$, for which the property*

$$c_{H(e)}(D|_{\mathbb{E}\setminus G} \biguplus H) < c_{D(e)}(D), \qquad \forall e \in G, \tag{1}$$

applies.

The equation (1) states that, at the point $D \in \mathbb{D}$ of the decision space, each of the economic agents from the set G can strictly decrease his individual transport costs by moving to the path $H(e)$ from the set $H(G)$, while all the other economic agents keep on using their current paths, as implied by the restriction $D|_{\mathbb{E}\setminus G}$ of the decision mapping D to the set $\mathbb{E} \setminus G$.

The strict inequality of respective unit path costs implies that an economic agent would not change to a new path, if, on the new path, he incurred the same transport costs as on his current path. Note that this implies somewhat conservative transport behaviour of network users. As observed in [4], this property, by itself, engenders very strong equilibrium tendencies in the dynamic behaviour of the market.

The cost function c, and the ability of agents to calculate its values, implicitly assumed by the above definition, embody the assumption of complete information. To be able to decide, which of the alternative paths he should use at a particular point of the time horizon, an economic agent must be, in the first place, familiar with the functional form of this function, and, in the second place, to be able to calculate its values, he must have a complete overview over the decisions of other economic agents.

As observed in [4], this assumption is not universally applicable in transport markets. These markets can often feature substantial numbers of economic agents, which would base their transport decisions on only incomplete information. This type of information would be, however, in modern transport systems provided by transport oriented applications of mobile communication devices. Such markets would be good approximations of the above idealized model.

Definition. *A point D_0 of the decision space $\mathbb{D}$ will be called a perfect equilibrium, if, at this point, no subset $G \subset \mathbb{E}$ of economic agents can improve their bargaining position. It will be called an equilibrium of nonnegative integer order $k < n$, if this condition is fulfilled only for subsets $G \subset \mathbb{E}$ including no more than k economic agents.*

According to this definition, a perfect equilibrium is also an equilibrium of any nonnegative integer order $k < n$.

Each point D of the decision space $\mathbb{D}$ determines a distribution of flows over the paths of the set $\mathbb{P}$ $f = (f_{p_1}, f_{p_2}, \dots, f_{p_m},) \in \mathbb{R}^m$, defined by

$$f_p = \sum_{e \in \mathbb{E}: D(e) = p} 1, \quad \forall p \in \mathbb{P}. \tag{2}$$

It will be called the flow distribution of D and denoted by $f(D)$. In this sense, we will speak also about *equilibrium distribution of flows*. Note that different decision mappings can lead to the same flow distribution: it can be that $f(D_1) = f(D_2)$ for $D_1 \neq D_2 \in \mathbb{D}$. Note also that the structure of our model allows only for integer path flow volumes.

3 Examples of Models

In this section, we will analyze several small examples of models to determine basic existence properties of the adapted equilibrium notion. We will restrict ourselves to additive linear models. In these models, the function c is defined by

$$c_p(D) = c_{0p} + \sum_{q \in P} c_{pq} \sum_{e \in \mathbb{E}: D(e) = p} 1, \qquad \forall p \in \mathbb{P}, D \in \mathbb{D}, \tag{3}$$

where the coefficients c_{pq} and initial values c_{0p} for $p, q \in \mathbb{P}$ are arbitrary real numbers. The number of services in the set $\mathbb{P}$ will be called the dimension of the model. Such a model can thus be given by a matrix

$$C = \begin{Vmatrix} c_{11} & c_{12} & \dots & c_{1m} \\ c_{21} & c_{22} & \dots & c_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mm} \end{Vmatrix}, \tag{4}$$

a vector of initial values $c_0 = (c_{01}, c_{02}, \dots, c_{0m})$, and the number n of economic agents using the network.

Definition. *We will say that a model with m paths and n economic agents and an additive linear cost function is of type $L(m, n)$. If c_0 is a zero vector, we will say that it is homogenous or of type $HL(m, n)$. In a similar way, we will use the notation $DL(m, n)$ and $DHL(m, n)$ for respective models with diagonal matrices.*

Unit transport cost functions with linear structure arise, for example, in modelling transport flows in urban networks, equipped with traffic lights. Consider a pair of intersecting road routes, whose intersection is equipped with traffic lights. The length of the time or the green light interval, for which the intersection is open to one or the other path, is a matter of traffic optimization schemes. In determining this time for both of the paths, such schemes would usually consider the amount of flow, using each of the paths. In the first approximation, this time could be, for example, taken to be proportional to the flows using each of the respective paths. If the time spent on the path itself is also taken to be, in the first approximation, proportional to its flow, summing up of this time and the time spent at a closed intersection (depending on the flow on the other path) would lead to a two-dimensional linear functional form of the above type. A dense urban network with a lot of intersections and public services can lead to quite complex higher dimensional linear structures of this type. This would be particularly true in a congested urban network, in which public services, such as bus and taxi services, can significantly affect each other and private car flows. These effects would be similar to those to be found in congested railway networks.

Linear cost structures can be useful also for different reasons. In the first place, using linear structures one can, in a simple way, model different levels of interdependencies between the different parts of the network or market observed. A matrix C with many zero or small coefficients can be, for example, a good model of only feebly and loosely interdependent parts of the transport market, while a matrix C with mostly nonzero coefficients would, in a similar way, represent a highly interconnected and interdependent structure of the respective transport market. Properties such as these can have interesting and important consequences for the global dynamic behaviour of the market.

In the second place, if problems with the existence of equilibria appear in the linear case, one can clearly expect similar or even more severe problems in models with nonlinear and/or discontinuous unit cost functions. In this sense, linear models are a good and reliable starting point for the analysis.

3.1 Model of type $L(2, n)$

A linear two-dimensional model can be given by a matrix

$$C = \begin{Vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{Vmatrix}, \tag{5}$$

and a vector $c_0 = (c_{01}, c_{02})$.

Owing to the small dimension of the model, a complete enumeration of inequalities, defining the equilibrium, is possible at least for sufficiently small numbers of economic agents n. Owing to the additive structure of these models, one can also work with simpler conditions for equilibrium flow distribution.

Denoting by $f^0 = (f_1^0, f_2^0)$ a perfect equilibrium flow distribution and, by f_{12}, and f_{21}, the amounts of flow that can move from the first path to the second one, and vice versa, in a perfect equilibrium, we must, in particular, have

$$\begin{aligned} c_{11}f_1^0 + c_{12}f_2^0 + c_{01} &\le c_{21}(f_1^0 - f_{12}) + c_{22}(f_2^0 + f_{12}) + c_{02} \quad \forall \quad 0 < f_{12} \le f_1^0, \\ c_{21}f_1^0 + c_{22}f_2^0 + c_{02} &\le c_{11}(f_1^0 + f_{21}) + c_{12}(f_2^0 - f_{21}) + c_{01} \quad \forall \quad 0 < f_{21} \le f_2^0. \end{aligned} \tag{6}$$

Rearranging these two inequalities, we get the following inequalities

$$\begin{aligned} f_{21}(c_{12} - c_{11}) + (c_{02} - c_{01}) \le f_1^0(c_{11} - c_{21}) + f_2^0(c_{12} - c_{22}) \le \\ \le f_{12}(c_{22} - c_{21}) + (c_{02} - c_{01}), \\ \forall \quad 0 < f_{12} \le f_1^0, \quad \forall \quad 0 < f_{21} \le f_2^0, \end{aligned} \tag{7}$$

which must be fulfilled by the equilibrium distribution f^0.

The inequalities (7) rather strongly bind flow distributions f^0 that can appear at an equilibrium point. For example: an all or nothing equilibrium flow distribution f_1^0 on the first path would have to satisfy also the inequality

$$f_1^0(c_{11} - c_{21}) \le f_{12}(c_{22} - c_{21}) + (c_{02} - c_{01}), \quad \forall \quad 0 < f_{12} \le f_1^0. \tag{8}$$

If the coefficients of the model are such that $c_{11} - c_{21} > 0$ and $c_{22} - c_{21} < 0$ and $c_{02} = c_{01}$, this would clearly be impossible as the flows f_1^0 and f_{12} can take on only nonnegative values. As a consequence, such a model would have no equilibrium of all or nothing type on the first path for any number of economic agents n. Using this approach, one can verify the equilibrium properties of the following examples of the two-dimensional models.

3.1.1 Example 1

The model of type $HL(2, n)$ with the matrix

$$C = \begin{Vmatrix} 19 & 13 \\ 23 & 17 \end{Vmatrix} \tag{9}$$

has one perfect equilibrium with the equilibrium flow distribution of $(0, 1)$ for $n = 1$ and no perfect equilibrium for any $n > 1$.

The example shows that, in linear models, the existence of integer equilibria is, in general, not guaranteed. The exception is, of course, the trivial case $n = 1$, for which, in a linear model with a single origin-destination pair, a perfect equilibrium always exists.

On the other hand, if, in the same model, we change the value of the coefficient c_{21} from 23 to -23, the model has at least one perfect equilibrium for any $n \in \mathbb{N}$. A model, such as this one, can, for example, describe intensive taxi services along a single route. The first path of the model would represent trips of users of taxi services, the second path trips of taxi drivers. The coefficient c_{11} would, in this case, describe the effects of longer waiting times for a taxi, when there are many potential users of taxi services around. The coefficient c_{12} would describe the effects of the congestion along the route, caused by increasing numbers of taxis, providing for the service. The negative coefficient c_{21} would describe the fact that taxi drivers get paid for each passenger transported, and the coefficient c_{22} would, like the coefficient c_{12}, describe the effects of the congestion on the costs of taxi drivers.

3.1.2 Example 2

The model of type $HL(2, n)$ with the matrix

$$C = \begin{Vmatrix} 7 & 13 \\ 5 & 11 \end{Vmatrix} \tag{10}$$

has a perfect equilibrium with the equilibrium flow distribution $(1, 0)$ for $n = 1$, a unique perfect equilibrium flow distribution $(2, 0)$ for $n = 2$, two perfect equilibrium flow distributions $(2, 1)$ and $(3, 0)$ for $n = 3$, and no perfect equilibria for any $n > 3$.

The example shows that the existence of perfect equilibria in a linear model depends on the number of agents involved. It also shows that the number of perfect equilibria, and the type of corresponding equilibrium flow distributions, can vary with n.

3.1.3 Example 3

The model of type $HL(2, n)$ with the matrix

$$C = \begin{Vmatrix} 3 & 5 \\ 2 & 11 \end{Vmatrix} \tag{11}$$

has a perfect equilibrium with the equilibrium flow distribution $(1, 0)$ for $n = 1$, a unique perfect equilibrium flow distribution $(2, 0)$ for $n = 2$, a unique perfect equilibrium flow distribution of the form $(n, 0)$ for $3 \leq n \leq 8$, two perfect equilibrium distributions $(8, 1)$ and $(9, 0)$ for $n = 9$, a unique perfect equilibrium distribution of the form $(n - 1, 1)$ for $10 \leq n \leq 16$, no perfect equilibrium for $n = 17$, a unique perfect equilibrium flow distribution of the form $(n-2, 2)$ for $18 \leq n \leq 23$, no perfect equilibrium for $24 \leq n \leq 26$, a unique perfect equilibrium flow distribution of the form $(n - 3, 3)$ for $27 \leq n \leq 30$, no perfect equilibrium for $31 \leq n \leq 35$, a unique perfect equilibrium flow distribution of the form $(n - 4, 4)$ for $36 \leq n \leq 37$, and no perfect equilibrium for $n > 37$.

The example shows that the existence of perfect equilibria in a linear model is far from being a trivial issue. It leads to sophisticated enough integer number problems.

From the economic point of view, this example shows that adding or subtracting a small number of economic agents can drastically change the dynamic behaviour of the market, as it can lead from a market with a perfect equilibrium to a market with no perfect equilibrium, or, vice versa, from a market with no perfect equilibrium to a market with a perfect equilibrium. In other words, such transitions can both cause the erratic dynamic behaviour of the market or, vice versa, help a market with such behaviour to calm down and regain a new perfect equilibrium. It is interesting to note that this property of traffic dynamics has been observed also using models of self driven many particle systems, e.g. [3].

3.1.4 Example 4

The model of type $HL(2, n)$ with the matrix

$$C = \begin{Vmatrix} 13 & 15 \\ 17 & 11 \end{Vmatrix} \tag{12}$$

has a unique perfect equilibrium distribution of the form $(0, n)$ for any $n \geq 1$. The example shows that well behaved nontrivial linear models do exist. The existence of equilibria follows from the existence theorem, given in the next section: a diagonal element is the smallest of matrix coefficients. Note that, in this model, the dependence of the unit path costs on the amount of flow on the other path is stronger than the dependence on the amount of flow on this path itself.

4 An Example of Existence Theorem

We are searching for integer solutions to the equilibrium set of inequalities. It is clear from the examples, presented in the previous section, that, for linear models, one cannot expect very general existence results, similar to those, which can be proved for first order equilibria of continuous models and models with incomplete information, e.g. [4].

For a general model of type $L(m, n)$, one can develop similar inequalities, characterizing the equilibrium flow distribution as in the two-dimensional case. However, the number of inequalities would clearly quickly explode with increasing values of m and n, as one must take account of all possible redistributions of flows among two or more paths of the set $\mathbb{P}$. It would be therefore difficult, if not impossible, to tackle a system with real world dimensions with presently available methods for finding integer solutions to systems of linear inequalities. One can, of course, still use smaller subsets of these inequalities to eliminate eventually impossible perfect equilibrium flow distributions.

On the other hand, one can tackle even examples with real world dimensions, if the model exhibits eventual additional properties, which can be exploited to prove the existence of its equilibria, or to yield them. In the sequel, we present a simple example of such an existence theorem.

Theorem (Trivial Lemma). *A linear model of type $L(m, 1)$ has at least one perfect equilibrium for any $m \in \mathbb{N}$.*

Proof: If the only agent in the model chooses the path p with the least value of $c_{pp} + c_{0p}$, this results in a perfect equilibrium. In the case that the value of this expression is the same for several paths, the model has multiple perfect equilibria. **Q.E.D.**

Theorem. *Assume that in a linear model of type $L(m, n)$ with the matrix C exists a path $p \in \mathbb{P}$, for which*

$$\begin{aligned} c_{pp} \leq c_{uv}, \qquad & \forall u, v \in \mathbb{P}, \\ c_{pp} + c_{0p} \leq c_{uu} + c_{0u}, \qquad & \forall u \in \mathbb{P} \end{aligned} \tag{13}$$

applies. Then the model has for all $n \in \mathbb{N}$ an all or nothing perfect equilibrium, in which all agents use the path p.

Proof: As the $c_{pp} + c_{0p}$ is the smallest of the sums of diagonal coefficients and respective initial values, the theorem applies for $n = 1$. Suppose that the model for n agents has an all or nothing perfect equilibrium, in which all agents use the path p. According to the definition of a perfect equilibrium, for any flow distribution $f = (f_v)_{v \in \mathbb{P}}$ with $\sum_{v \in \mathbb{P}} f_v = n$, there must be a path $u \in \mathbb{P}$ with nonzero flow $f_u \neq 0$, for which

$$c_{pp} n + c_{0p} \leq \sum_{v \in \mathbb{P}} c_{uv} f_v + c_{0u}. \tag{14}$$

The distribution f can be adjusted with an additional economic agent by placing this agent on any of the paths from the set $\mathbb{P}$. Let us assume that we place him on the path v_1. Adding to the inequality for u, the inequality $c_{pp} \leq c_{uv_1}$, we get

$$c_{pp}(n + 1) + c_{0p} \leq \sum_{v \in \mathbb{P}: v \neq v_1} c_{uv} f_v + c_{uv_1}(f_{v_1} + 1) + c_{0u}. \tag{15}$$

By doing this for all $v_1 \in \mathbb{P}$ and all possible flow distributions f of n agents, we get all possible distributions of $n + 1$ agents on the paths of the set $\mathbb{P}$. That means that, in this way, we can produce all the inequalities that define an all or nothing perfect equilibrium flow distribution of $n + 1$ agents on path p. This proves the theorem by induction on n. **Q.E.D.**

Theorem (On Diagonal Model). *A diagonal linear model of type $DL(m, n)$ has at least one perfect equilibrium for any $m \in \mathbb{N}$ and any $n \in \mathbb{N}$.*

Proof: in a diagonal model, one can apply a similar argument as in the proof of the preceding theorem. If $(f_1^0, ..., f_n^0)$ is a perfect equilibrium distribution for n agents, placing an additional agent on the path with the least value of $c_k(f_k^0 + 1)$ produces a perfect equilibrium for $n + 1$ agents. **Q.E.D.**

These two theorems are the reminiscence of the well known property of transport markets: when a new, technologically and economically advanced, technology arrives, these markets often tend to adopt the new technology at the expense of older ones.

The condition (13) is sufficient for the existence of a perfect equilibrium in a linear model, but it is not a necessary one. This is evident from the equilibrium properties of the model (9), in which the value of the coefficient $c_{21} = 23$ is changed to $c_{21} = -23$. In spite of the fact that, in this case, a non-diagonal coefficient c_{21} is the smallest coefficient of the adapted matrix, the adapted model has a perfect equilibrium for any $n \in \mathbb{N}$.

5 Economic and Dynamic Considerations

Synchronized or unsynchronized decisions of economic agents will, along the time horizon, describe a trajectory in the decision space $\mathbb{D}$. A perfect equilibrium would be a fixed point of this dynamic process. It would represent a point of the trajectory, at which all economic agents would start to repeat their path decisions, as they would have no economic initiative to change their decisions.

In the perfect equilibrium, the functioning of the market does not contain any economic inefficiencies, which could be with benefit exploited either by individual economic agents, or by simultaneous actions of groups of agents. All such inefficiencies would have been effectively cleared off from the market during the equilibrium process.

In an equilibrium of order k, the functioning of the market still contains economic inefficiencies, which can be exploited by simultaneous actions of groups of economic agents containing more than k agents.

Note that equilibria of order 1 are, in fact, user equilibria, as defined by the traditional approach. These equilibria can be interpreted also as equilibria of users with little or no predictive capabilities. Economic agents with greater predictive capabilities would eventually start to repeat their decisions in equilibria of higher orders. In this sense, a perfect equilibrium can be interpreted also as that point of the decision space, at which economic agents exhaust all of the possibilities for improving their bargaining positions by improved predictions of the underlying equilibrium process. At this point, their predictive capabilities become economically superfluous, as all of the economic agents start to repeat their path choices, and the global behaviour of the market becomes predictable.

As the probability of successful simultaneous actions of k economic agents decreases with increasing of the value of k, equilibria of higher orders would have a tendency to persist for longer periods of time along the trajectory. As such, these equilibria would also appear as natural temporary convergence points of algorithms of greedy type, which have been traditionally used in constructing equilibria of first order in practical applications.

These algorithms usually consist of passes, in which all economic agents, in a sequential or parallel manner, try to improve their bargaining positions. It is well known that such algorithms are usually very greedy indeed.

It can be, for example, easily verified by inspecting the algorithmic behaviour of a large number of randomly generated linear models, discussed in this paper, that the number of passes, needed to arrive at an equilibrium of order 1 only very slightly depends on the cardinality of the sets of economic agents and paths involved. In models with many nonzero coefficients (implying a highly interconnected functioning of the market) such an algorithm, in average, requires no more than 2 to 3 passes to reach an equilibrium of order 1 (if it, of course, exists in observed models). In other words, under the assumption of complete information, and for most practical purposes, the length of the path to an equilibrium point of first order does not depend on the cardinality of the sets $\mathbb{P}$ and $\mathbb{E}$.

This is in contrast with the algorithmic behaviour of models, based on incomplete information. It was shown in [4] that, under the assumption of incomplete information, equilibria of first order always exist (under literally no conditions for unit transport cost functions). It was also shown that, in such models, one can with high probability, or, in other words, for all practical purposes, expect that the average number of passes needed to achieve an equilibrium point of first order would be a low integer multiple of the cardinality of the path set $\mathbb{P}$. Though the number of alternative services, considered by individual economic agents, would be usually relatively small, this, nevertheless, implies that, under the assumption of incomplete information, the length of this path would be, as a rule, substantially longer than under the assumption of complete information.

Concerning this issue, it is important to observe that a short average path to equilibrium is, by itself, of crucial importance for global economic efficiency of the respective market. It implies that such a market can, if disturbed by events in its environment, or in the market itself, quickly regain its equilibrium. This would, in turn, increase the probability that it would, at any point of the time horizon, function in, or near equilibrium, allowing for reasonably efficient use of its resources.

The difference between the length of the path to equilibrium in models with complete and incomplete information can thus be seen as one of the main properties determining the global efficiency of the non-cooperative markets with complete information. It is also clear from this difference that the efficiency of non-cooperative markets stands and falls with the availability and reliability of information flows, which ultimately enable the economic agents to make informed decisions. The absence of such information flows and/or their low quality would, as a rule, lead to the persistence of various market inefficiencies, which would, in turn, lead to low global efficiency of the market.

The corresponding algorithms for construction of equilibria of higher orders have not yet attracted the attention of researchers, so, at this point, it is difficult to comment on algorithmic properties of these equilibria. When, in the above algorithms, one substitutes groups of agents for individual agents, one can, in principle, expect the same greedy behaviour of the resulting procedures for higher order equilibria. However, the length of individual steps, which would in an algorithm for equilibrium of, say, order k, theoretically require complete enumeration of all possible groups of k agents, would, in real world dimensions, clearly quickly increase beyond the capabilities of present day computers. At the same time, and with increasing k, the optimization problems, to be solved by individual groups of users, also increase in difficulty and size.

Concerning this issue, one first has to realize that real world markets possess enormous capabilities for parallel and sequential processing of information, allowing for real world implementation of quite complex decision algorithms of this type. It has been often observed that it is better to leave the markets themselves to sort out eventual problems than try to solve or to alleviate them using measures, whose structure would, in view of these capabilities, only poorly match the intricacies and processing powers of market dynamics. It is also important to observe that real world markets have, at their disposal, something that computer applications often lack: a lot of time. Given enough of time, the markets can be expected to exploit even economic inefficiencies, whose clearing off the market would be otherwise rather improbable. This, of course, ultimately depends also on the processing and predictive capabilities and enterprising spirit of economic agents involved.

In the second place, it is reasonable to expect that, in real world situations, and in planning their moves, individual economic agents consider only one, or, at the very best, only a small number of subgroups of agents, for which they actually carry out the necessary calculations or approximations of respective values of cost functions. In other words, in individual passes of the algorithms for constructing higher order equilibria, one could basically work only with sets of subgroups, forming suitable set coverings of the complete set of economic agents. This would substantially reduce the length of individuals passes. However, the average number of passes of such an algorithm, required to reach the equilibrium, and possible strategies for picking up of alternative coverings in consecutive passes of such an algorithm, are topics, which, at least to the knowledge of the authors of this paper, have not been researched so far.

These observations would, of course, not apply for a market with no perfect equilibrium.

The dynamics of the decision making process, carried out by its economic agents, would, in this case, force the market to move along more or less complex trajectories through the decision space, containing only out of equilibrium points. Such trajectories would lead either to equilibria of smaller orders, if these eventually existed, or to more or less complex types of cycles, typically observed in this type of discrete situations. In functioning of such a market, at least some level of economic inefficiencies would therefore always be present. As a consequence, for such a market, one could not apply the well known hypothesis of efficient markets, e.g. [7], which is usually assumed and discussed in connection with the dynamic behaviour of capital markets.

There are, of course, significant differences between the equilibrium processes to be found in capital and transport markets; at the capital markets, they are part of the process of determination of prices, which occurs, in a centralized way, at stock markets, and are, as such, and to a high degree, almost instantaneous; in transport markets, they involve, besides the economic agents themselves, also various governmental and nongovernmental institutions and can take up substantial periods of time along the time horizon. In transport markets, they can be, in fact, often an ever going process.

There are, nevertheless, some additional similarities between these two type of markets. For example, during the equilibrium process itself, the economic agents do not know the exact total costs, connected with using a particular transport service, until the very last moment that they actually consume it. As during this process, these costs depend on the combined decisions of all the agents, they would, in fact, tend to exhibit similar random behaviour as prices at stock markets, though possibly of more modest and more predictable random nature.

At the end, it must be observed that the dynamics of transport equilibrium processes is, by their very nature, and in spite of eventual deterministic cost function assumed, of non-unique type. This is the consequence of the situations, in which economic agents must choose between two or more alternative services with equal current costs. The only plausible resolution of these situations is to assume that the resulting choice of a service is a matter of pure chance. It is important to observe that the same type of situations can also appear during the group behaviour, when economic agents choose among various alternative subgroups of economic agents, which they will use in estimating future transport costs. This would clearly additionally increase the level of non-uniqueness, present in the global dynamic behaviour of the market, and, eventually lead to dynamic trajectories of transport equilibrium processes of highly unpredictable nature.

In this respect, it is important to note that these nonunique situations can, in particular, turn the cycling types of trajectories into chaotic type of dynamic trajectories, leading to the chaotic type of global behaviour of the market.

6 Summary and Outlook

Adapting the traditional notion of user equilibrium of network flows in a way to allow for the group behaviour of economic agents leads to quite sizeable systems of inequalities, defining the equilibrium state. In models with real world dimensions, they can be expected to be efficiently solved only in the presence of eventual additional regularities and patterns of the functional relationship between unit transport costs and network flow volumes.

This, by itself, does not necessarily diminish the relevance of this equilibrium notion for real world transport networks and markets, as they possess substantial capabilities for parallel and sequential processing of information. These capabilities can lead to sophisticated enough real world "algorithms" for reaching equilibria of this type.

The existence of perfect equilibria, in particular, depends on the number of economic agents, competing for the network resources. This can produce discrete gaps in the number of economic agents, for which perfect equilibria do not exist. Incremental changes of the demand volumes can, in the presence of such gaps, lead not only to calming down eventual erratic dynamic behaviour of the market, but also to causing such behaviour. This is just one more proof of considerable dynamic intricacies of equilibrium behaviour of transport networks and markets.

Further research of this notion should be directed at special cases of unit transport cost functions, particularly of linear type, that would guarantee the existence of perfect equilibria and/or allow for an efficient solution of the corresponding equilibrium system of inequalities.

Important insights into the equilibrium dynamic of transport markets could be also provided by the study of efficient parallel or sequential algorithms for constructing equilibria of higher orders.

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References

[1] Beckman, M. & Mcguire, C. & Winston, B., 1956, *Studies in economics of transportation*, (Yale University Press, New Haven).

[2] Daganzo, C. & Sheffi, Y., 1977, On the stochastic models of traffic assignment, *Transportation Science* **11**, 253-274.

[3] Helbing, D. & Schönhof, M.& Kern, D., 2002, Volatile decision dynamics: experiments, stochastic description, intermittency control and traffic optimization, *New Journal of Physics* **4**, 33.1-33.6.

[4] Sever, V.,1993, Dynamic equilibria in transport systems, *European Journal of Operational Research* **71**, 419-438.

[5] Smith, M.J., 1979, The existence, uniqueness and stability of traffic equilibria, *Transportation Research* **13B**, 295-304.

[6] Wardrop, J.G., 1952, Some theoretical aspects of road traffic research, *Proc. Inst. Civil Eng.* **Part II**, 225-378.

[7] Fama, E., 1976, *Foundations of Finance*, (Basic Books, Chicago).

